Vol. 29, No. 5, May-June, 1956

MATHEMATICS magazine

MATHEMATICS MAGAZINE

Formerly National Mathematics Magazine, founded by S.T. Sanders.

Address editorial correspondence to Glenn James, special papers to the editors of the departments for which they are intended, and general papers to a member of the executive committee.

Manuscripts should be typed on 8½" x 11" paper, double-spaced with 1" margins. We prefer that, in technical papers, the usual introduction be preceded by a *Foreword* which states in simple terms what the paper is about.

The Mathematics Magazine is published at Pacoima, California by the managing editor, bi-monthly except July-August. Ordinary subscriptions are 1 yr. $3.00; 2 yrs. $5.75; 3 yrs. $8.50; 4 yrs. $11.00; 5 yrs. $13.00. Sponsoring subscriptions are $10.00; single copies 65¢. Reprints, bound ½¢ per page plus 10¢ each, (thus 25 ten page reprints would cost $1.25 plus $2.50 or $3.75) provided your order is placed before your article goes to press.

Subscriptions and related correspondence should be sent to Inez James, 14068 Van Nuys Blvd., Pacoima, California.

Advertisers should contact The Lees Agencies, P.O. Box 906, Hollywood 28 California.

Entered as second-class matter March 23. 1948 at the Post Office, Pacoima, Calif. under act of Congress of March 8, 1876.

MATHEMATICS MAGAZINE

VOL. 29, No. 5, May - June 1956

CONTENTS

OUR CONTRIBUTORS

B. J. Pettis, Professor of Mathematics, Tulane University, was born in Pittsburgh, Pennsylvania in 1913. He received his formal education at Wofford College (A.B. '32), at the University of North Carolina (M.A. '33) and at the University of Virginia (Ph.D. '37) where he worked with Professor E. J. McShane. During the next few years he did research and teaching at Harvard and Yale Universities. From 1941-45 he served in the U.S. Army in the South Pacific. He has been at Tulane since 1947.

James A. Ward, Professor of Mathematics, University of Kentucky, was born in South Carolina in 1910. A graduate of Davidson College (B.A. '31), Louisiana State University (M.S. '34) and the University of Wisconsin (Ph.D. '39), he has taught at Davidson College, Delta State Teacher's College and at the University of Georgia. He spent almost four years as a Naval Officer, half of it teaching at the U.S. Naval Academy. At present he is on a leave of abscence from Kentucky, at Holloman Air Base, New Mexico, as a consultant mathematician.

Murray R. Spiegel is a graduate of Brooklyn College, and received his M.S. in Theoretical Physics and his Ph.D. in applied mathematics from Cornell University. He is now Associate Professor of Mathematics at the Hartford Graduate Center which has recently been established by Rennselaer Polytechnic Institute at Hartford, Connecticut.

Ali R. Amir-Moéz was born in Teheran, Iran in 1916. He received the degree of Licencie de Science in June 1943 from the University of Teheren, and was an Instructor of Mathematics, Teheran Technical College from 1943-47. After coming to this country in 1947, he studied at Springfield College during 1947-48, and at the University of California, Berkeley and Los Angeles from 1948-55 and received his Ph.D. in Mathematics at U.C.L.A. in 1955. During 1954-55 he taught at Santa Barbara College, University of California, and new is an Instructor of Mathematics at the University of Idaho.

Bancroft H. Brown, B.P. Cheney Professor of Mathematics, Dartmouth College, was born in Hyde Park, Mass. in 1894. A graduate of Brown University (A.B. '16, A.M. '17), he served in the Medical Corps of the U.S. Army from 1917-19 and then resumed his graduate studies at Harvard University (Ph.D. '22). He has taught at Dartmouth ever since, and has served as Department Chairman, as well as Chairman of the Division of Sciences there. He has also been President of the New Hampshire Academy of Sciences and Vice-President of the Mathematical Association of America.

(Continued on page 268)

(Continued from back of CONTENTS)

Harold F. Mathis, Research Engineer, Goodyear Aircraft Corp., Akron, Ohio, was born at Wichita Falls, Texas in 1916. He was educated at the University of Oklahoma (B.S. '39, E.E. '55), Texas A. & M. College (M.S. '41, E.E. '52), and Northwestern University (Ph.D. '53). Prior to his present position, he was an Associate Professor of Electrical Engineering at the University of Oklahoma ('49-'54). His major interest is the applied mathematics associated with radar antennas.

Articles by the following authors appeared in the last two issues.

Dr. Adrien L. Hess, Associate Professor of Mathematics at Montana State College, is in charge of the training of teachers of elementary and secondary mathematics. He holds degrees from Missouri Valley College, Central Missouri State College, Montana State University and the University of Wisconsin. He taught mathematics and science for 14 years in high schools of Montana before joining the Mathematics staff at Montana State College.

Robert Steinberg, Assistant Professor of Mathematics, University of California, Los Angeles, was born in Stykon, Roumania in 1922. He received the Ph.D. from the University of Toronto in 1948, and joined the U.C.L.A. faculty that same year. He specializes in the theory of group representations.

Walter R. Van Voorhis, Professor of Mathematics and Department Chairman, Fenn College, is a graduate of Pennsylvania State University (B.A. '30; M.A. '31; Ph.D. '41). From 1931-41 he was Assistant Professor of Mathematics at Penn. State and served as Director Schuylkill Undergraduate Branch of the College. During World War II, Dr. Van Voorhis served as Deputy Chief, AAF Reports Control Branch, Statistical Division, Washington, D.C. He specializes in Mathematical statistics, and is the co-author of several text books on statistics and college mathematics.

for two study groups, or $40 for one laboratory and one or more study groups. General sessions are open to any enrollee, irrespective of the fee paid.

The purpose of this conference is to bring together teachers interested in mathematics - arithmetic through calculus - to study common problems in the teaching of mathematics and to learn new uses of mathematics in various fields of endeavor. Principal instructors, in addition to Professor Bell will be Ida May Bernhard, Consultant in Secondary Education, Texas Education Agency, Austin, Texas; Frances Ceccarini, of the Burbank City Schools, Burbank, California; Marion Cliffe of the Los Angeles Board of Education, Los Angeles, California; William Glenn, Assistant Principal of John Muir High School in Pasadena, California; and L. Clark Lay of Pasadena City College and lecturer in mathematics at U.C.L.A.

Applications for enrollment or requests for additional information should be addressed to University of California Extension, Mathematics, Los Angeles 24, California.

* * * * * * * * * * * * *

Civil Service Positions

There is an urgent need for Chemists, Mathematicians, Metallurgists, Physicists, and Electronic Scientists in the Washington, D. C., area, the United States Civil Service Commission has announced. Vacancies are in various Federal agencies and pay salaries ranging from $4,345 to $11,610 a year.

To qualify for positions paying $4,345 a year, applicants must have had appropriate education of a combination of education and experience. For the position of Electronic Scientist, appropriate technical or scientific experience alone may be qualifying. For higher-grade positions, professional experience is also required. Graduate study may be substituted for all or part of this experience, depending on the grade of position. No written test is required.

Further information and application forms may be obtained at many post offices throughout the country, or by writing to the U.S. Civil Service Commission, Washington 25, D.C. Applicants should ask for Announcement No. 45(B). Applications will be accepted by the Board of U.S. Civil Service Examiners, National Bureau of Standards, Washington 25, D.C., until further notice.

* * * * * * * * * * * * *

SEPARATION THEOREMS FOR CONVEX SETS

B. J. Pettis *

The following pages represent an attempt to present a fairly simple exposition of some of the well known basic results on convex sets in real linear spaces having topologies. The results discussed here are those that naturally group themselves around the separation theorem (Theorem 3) and include several of its standard consequences concerning the geometry of convex sets, the Minkowski-Price-Krein-Milman theorem on extreme points, and a version of the so-called fundamental theorem of game theory. The only previous knowledge required here consists essentially of an acquaintance with elementary maneuvers in real linear spaces and in topological spaces; beyond these the Hahn-Banach theorem and the Hausdorff Maximality Principle are each invoked once, and some standard criteria for compactness are called upon occasionally.

Specifically we shall be concerned with convex and midpoint convex sets in a real linear space having a topology such that addition and scalar multiplication are continuous in each variable separately. Such spaces have been considered by Nikodym [13] and Klee [8] and have the advantage that theorems stated in such a context always have as corollaries purely algebraic statements about arbitrary real linear spaces. For in any such space E there is a certain natural topology, called the "core topology" by Klee [8] and sometimes the "radial topology" by others, in which a subset A of E is open if and only if for each x in A and each y in E there is some positive real number $\epsilon_{x,y}$ such that $x + \lambda y$ is in A whenever $|\lambda| < \epsilon_{x,y}$. With respect to this topology addition and scalar multiplication are continuous in each variable, so that all theorems and corollaries stated below except Corollary 3.13 are true in any real linear space E with respect to this topology. More will be said on this at the end of the paper.

The histories of the results stated here are so long and involved that no attempt has been made to describe them. We assert only that almost all of these theorems in their present forms began specifically with Minkowski [12] in his work on convex sets in Euclidean 3-space (Price [14]), and that this is particularly true of the chief separation theorem (Theorem 3) and the theorem on extreme points (Theorem 4). In view of the present importance of these two results it might, however, be in order to recall briefly that the former was extended to separable Banach Spaces by Ascoli, to normal linear spaces by Mazur,

---

* This paper was written under the provisions of Contract N7-onr-434, Task Order III, Navy Department, Office of Naval Research.

Eidelheit, Kaukutani, and Tukey, and to its present generality by Klee [8] using an argument of Stone's [15]. The second was extended to general Banach spaces by G.B.Price [14] (it is well known that Price's assumption of strict convexity for the norm is no actual restriction) and to the adjoint of a Banach space with the weak-* topology by Krein and Milman [10]. The present version was stated by Artin and J. L. Kelley [7].

Let E be any linear space over the scalar field S, where S is the real number system. Suppose that θ denotes the zero element of E and that all other lower case letters represent elements of S. The symbol $A\setminus B$ will denote the set $\{x\colon x \in A,\ x \notin B\}$. Suppose also that E is a topological space, in which the closure and interior of any subset A are denoted by A^* and A^0 respectively, and that the algebra and the topology in E are related as follows: the function $x + \alpha y$, where $x \in E$, $y \in E$, and $\alpha \in S$, is continuous in each of the three variables whenever the other two are fixed. This is equivalent (in view of the commutativity of addition) to assuming that $x+y$ is continuous in y for each x and that αy is continuous in y for each α and continuous in α for each y. If given $A \subset E$, $B \subset E$, and $\alpha \in S$ we henceforth write $A + B$ for the set $\{x + y\colon x \in A,\ y \in B\}$ and αA for the set $\{\alpha x\colon x \in A\}$, these continuity conditions imply (actually are equivalent to) the following, as can easily be verified: for any $A \subset E$ the equalities $x + A^0 = (x + A)^0 = (A + x)^0 = A^0 + x$ and $x + A^* = (x + A)^* = (A + x)^* = A^* + x$ hold for any x in E; the equalities $\alpha A^0 = (\alpha A)^0$ and $\alpha A^* = (\alpha A)^*$ are true for each non-zero α in S; and for each non-zero element x in E the set $\{\beta\colon \beta x \in A^0\}$ is open in S and the set $\{\gamma\colon \gamma x \in A^*\}$ is closed in S. We shall require one particular computational rule: for any two sets A and B in E, $A^* + B^* \subset (A + B)^*$. The proof is briefly as follows. If $a \in A$ then $a + B^* = (a + B)^* \subset (A + B)^*$, so that $A + B^* \subset (A+B)^*$; by commutativity $A^* + B \subset (A + B)^*$, also holds, and hence $A^* + B^* \subset (A + B^*)^* \subset ((A + B)^*)^* = (A + B)^*$.

A subset A of E is *convex* if $\lambda x + (1 - \lambda)y \in A$ whenever $x \in A$, $y \in A$, and $0 \leq \lambda \leq 1$, i.e., if $\lambda A + (1 - \lambda)A \subset A$ whenever $0 \leq \lambda \leq 1$; it is *midpoint convex* if these relations hold whenever $\lambda = \frac{1}{2}$, that is, if $A + A \subset 2A$. The intersection of any number of (midpoint) convex sets is again (midpoint) convex, and $A + B$ and λA are (midpoint) convex whenever A and B are and λ is any element of S. If A is midpoint convex it can be established by induction that $\lambda A + (1 - \lambda)A \subset A$ whenever λ is a dyadic rational $k/2^n$, where k and n are non-negative integers and $k \leq 2^n$. A (midpoint) convex set is a *(midpoint) convex body* if its interior is non-void. The intersection of all convex sets containing a given set A is convex by a remark above, and is clearly the least convex set containing A; this will be denoted by $C(A)$ and will be called the *convex cover of* A.

Theorem 1. *If $A + A \subset 2A$ then A^* is convex; hence A^* is convex if A is midpoint convex.*

We first note that A^* is midpoint convex, since $A^* + A^* \subset (A + A)^* \subset (2A^*)^* = 2(A^*)^* = 2A^*$. Now let a and b be any two points in A^* and consider the map $\phi(\lambda) = \lambda a + (1 - \lambda)b = b + \lambda(a - b)$. From our continuity assumptions ϕ is continuous on the reals to E and hence $\phi^{-1}(A^*)$ must be closed. This set clearly contains 0 and 1, and is easily seen to be midpoint convex since A^* is. Obviously such a set must include the whole interval $[0,1]$ in the reals, and so $\lambda a + (1 - \lambda)b$ is in A^* whenever $0 \leq \lambda \leq 1$. Thus A^* is convex.

This implies that if A is any set in E the closure, $C(A)^*$, of its convex cover is the least of the closed convex sets containing A; for by the above theorem $C(A)^*$ is such a set, and clearly any such set must contain $C(A)$ and hence the closure of $C(A)$.

An essential notion in any discussion of convex sets is that of the Minkowski functional p_B of any set B, defined on E to $[0,\infty]$ by $p_B(x) = \inf\{\lambda: \lambda \text{ a dyadic rational}, \lambda > 0, x \in \lambda B\}$, where the infimum of the void set is taken to be $+\infty$. Note that $p_B(\mu x) = \mu p_B(x)$ for any x and any positive dyadic rational μ. If $\theta \in B$ then $p_B(\theta) = 0$ and hence $p_B(0x) = 0 \cdot p_B(x)$ whenever $p_B(x) < \infty$.

Theorem 2. *Suppose B is midpoint convex and $\theta \in B$. Then (i) p_B is real-valued, (ii) $p_B(x + y) \leq p_B(x) + p_B(y)$ for any x and y in E, (iii) p_B is uniformly continuous in E, and (iv) $p_B(\lambda x) = \lambda p_B(x)$ for any x in E and any real $\lambda \geq 0$, and (v) the sets $C = \{x: x \in E, p_B(x) < 1\}$ and $D = \{x: x \in E, p_B(x) \leq 1$ are non-void and convex, $C = B^\circ = (B^*)^\circ$, and $D = B^* = (B^\circ)^*$.*

Since $\theta \in B^\circ$ and λx is continuous in λ, it follows for each x in E that $\lambda x \in B$ for all sufficiently small λ. Hence $x \in \lambda^{-1}B$ for at least one $\lambda > 0$, so that $p_B(x) < \infty$ for each x in E, establishing (*i*). Regarding (*ii*), suppose x and y are any two points in E and $x \in \lambda B$ and $y \in \mu B$ where λ and μ are positive dyadic rationals. Then $x + y \in \lambda B + \mu B = (\lambda + \mu)[\rho B + (1 - \rho)B]$ where ρ is the positive dyadic rational $\lambda/(\lambda + \mu)$. Since $\rho B + (1 - \rho)B \subset B$ due to the mid-point convexity of B, we have $x + y \in (\lambda + \mu)B$ and so $p_B(x + y) \leq \lambda + \mu$. From this and the definition of p_B it is evident that $p_B(x + y) \leq p_B(x) + p_B(y)$.

To establish the uniform continuity of p_B, let δ be any positive dyadic rational and set $V_\delta = U \cap (-U)$ where $U = (\delta B)^\circ = \delta B^\circ$. Clearly V_δ is an open set containing θ, and we have only to show that $x - y$ being in V_δ implies that $|p_B(x) - p_B(y)| \leq \delta$. If $x - y \in V_\delta$ then $x - y \in \delta B$ and so $p_B(x - y) \leq \delta$; and since $V_\delta = -V_\delta$, $y - x$ is also in V_δ and so $p_B(y - x) \leq \delta$. But $p_B(x) - p_B(y) = p_B((x - y) + y) - p_B(y) \leq p_B(x - y) + p_B(y) - p_B(y) = p_B(x - y) \leq \delta$, and similarly $p_B(y) - p_B(x) \leq p_B(y - x) \leq \delta$. Thus $|p_B(x) - p_B(y)| \leq \delta$.

In particular, then, p_B is continuous. Hence $p_B(\lambda x)$ is, for any fixed x, a continuous function of λ on $[0,\infty]$ to the reals, and so is $\lambda p_B(x)$. These two continuous functions coincide on the dense set of all positive dyadic rationals and hence must coincide on all of $[0,\infty]$, proving (iv).

Finally, the sets C and D are obviously non-void, and are easily seen to be convex in view of (ii) and (iv). Clearly $B \subset D$, and since $\lambda B = \lambda B + (1-\lambda)\theta \subset \lambda B + (1-\lambda)B \subset B$ for any positive dyadic rational $\lambda < 1$, it follows that $C \subset B$. The continuity of p_B implying that C is open and D is closed, we have $C \subset B^\circ \subset B \subset B^* \subset D$, so that $C \subset B^\circ \subset D^\circ$ and $C^* \subset B^* \subset D$. If it is shown that $D^\circ \subset C$ and $D \subset C^*$ then clearly $C = B^\circ = D^\circ$ and $D = B^* = C^*$, so that $C = B^\circ = (B^*)$ and $D = B^* = (B^\circ)^*$. To see that $D^\circ \subset C$ fix x in D°; since λx is continuous in λ and D° is open, there is some $\mu > 1$ such that $\mu x \in D^\circ$ and hence $p_B(\mu x) \leq 1$, so that $p_B(x) \leq 1/\mu < 1$, and hence $x \in C$. Finally, if $x \in D$ then $p_B(x) \leq 1$, and on setting $x_n = \mu_n x$ where $\mu_n = 1 - 2^{-n}$, it follows that $x_n \in C$, since $p_B(x_n) = \mu_n < 1$, and that $\lim_n x_n = x$, so that $x \in C^*$. This completes the proof of Theorem 2.

Now suppose that A is any midpoint convex set in E. If A° is void it is obviously convex; if A° is not void choose a in A° and set $B = A - a$. Then B satisfies the hypotheses of the theorem and so B° is convex. Since $B^\circ = (A-a)^\circ = A^\circ - a$, clearly A° is convex. Moreover $(B^*)^\circ = B^\circ$, and hence $((A-a)^*)^\circ = (A-a)^\circ$, so that $(A^* - a)^\circ = A^\circ$ a, and thus $(A^*)^\circ - a = A^\circ - a$, whence $(A^*)^\circ = A^\circ$. Similarly, $(A^\circ)^* = A^*$. Thus we have

Corollary 2.1 *If A is midpoint convex A° is convex; if also A° is non-void then $(A^*)^\circ = A^\circ$ and $(A^\circ)^* = A^*$.*

Thus if A is any set in E the set $C(A)^\circ$ is convex; if also A is open then $A \subset C(A)^\circ$, and so $C(A) \subset C(A)^\circ$ by definition of $C(A)$. This leads us to

Corollary 2.2 *If A is an open set in E so is $C(A)$.*

The final corollary to Theorem 2 is frequently useful but will not be used in the remainder of the paper; it is an easy consequence of part (v) of the theorem and a simple translation argument.

Corollary 2.3 *Let A be midpoint convex and $a \in A^\circ$. Set $B = A - a$. Then for any point x in E the following are true: (1) if $p_B(x) = 0$ then $a + \lambda x \in A^\circ$ for every $\lambda \geq 0$; (2) if $p_B(x) > 0$ let $\beta = 1/p_B(x)$; then $a + \lambda x \in A^\circ$ for $0 \leq \lambda < \beta$, $a + \beta x \in A^* \setminus A^\circ$, and $a + \lambda x \in E \setminus A^*$ for $\beta < \lambda$.*

A *functional* is any real-valued function f on E; it is *additive* if $f(x+y) = f(x) + f(y)$ for every x and y in E. Any additive functional has the property that $f(\lambda x) = \lambda f(x)$ for any x in E and any rational λ; if f is additive and this equality holds for every real λ, f is described as *linear*. The set of all linear functionals on E will be denoted by E', and the set of all continuous linear functionals by $\widehat{E}$; the set of all those not identically zero will be denoted by $E'\setminus 0$ and $\widehat{E}\setminus 0$ respectively. From a simple translation argument it follows that if f is additive and continuous at θ it is continuous; and if f is additive and continuous it is linear, since for any fixed x the functions $f(\lambda x)$ and $\lambda f(x)$ coincide on the dense set of all rational λ's and are continuous on the reals and hence coincide for all real λ.

When f is any functional on E and A is any subset of E the symbol $f(A)$ will represent the set $\{f(x): x \in A\}$, and $f(A)^{\circ}$ and $f(A)^*$ the interior and closure, respectively, of $f(A)$.

Lemma 1. *Let A be an open set in E and let f be an element of $E'\setminus 0$. Then $f(A)$ is open in the reals.*

Choose b in E such that $f(b) \neq 0$. Let a be any element of A. Since $\phi(\lambda) = a + \lambda b$ is continuous and $\phi(0)$ is in A, there is some $\delta > 0$ such that $\phi(\lambda) \in A$ for $|\lambda| < \delta$. Hence $f(a) + \lambda f(b) = f(a + \lambda b) \in f(A)$ for such λ, and since $f(b) \neq 0$ the set $\{f(a) + \lambda f(b): |\lambda| < \delta\}$ is open about $f(a)$. Thus $f(A)$ is open.

Before drawing from Lemma 1 a corollary, let us recall that if M is any subset of a topological space T and f is a continuous real function on T then $\inf f(M) = \inf f(M^*)$ and $\sup f(M) = \sup f(M^*)$.

Corollary. *If $f \in \widehat{E}\setminus 0$ and A is a midpoint convex body in E, let $\alpha = \inf f(A)$ and $\beta = \sup f(A)$. Then $\alpha < f(x) < \beta$ for any x in A°, and $\inf f(A^{\circ}) = \inf f(A^*) = \alpha < \beta = \sup f(A^*) = \sup f(A^{\circ})$.*

By the lemma $f(A^{\circ})$ is an open set of reals; since it is non-void (due to A° being so), clearly $\alpha < f(x) < \beta$ for any x in A°, and hence $\alpha < \beta$. From the remark preceding the corollary, $\inf f(A^{\circ}) = \inf f((A^{\circ})^*)$ and $\inf f(A) = \inf f(A^*)$; since $(A^{\circ})^* = A^*$ by Corollary 2.1, clearly $\inf f(A^{\circ}) = \inf f(A^*) = \alpha$; similarly, $\beta = \sup f(A^*) - \sup f(A^{\circ})$.

Lemma 2. *Let f be any additive functional on E that is bounded above (or below) on some non-void open set G. Then (i) f is continuous and linear; (ii) either f is constant on G, in which case f vanishes identically, or else $f(G)$ is a non-void open set of reals that is bounded above (or bounded below).*

On choosing x_0 in G the set $G - x_0$ is open and contains θ, and the additive functional f is bounded on this set in the same fashion as on G; thus we may assume G to contain θ. From an earlier remark preceding Lemma 1 conclusion (i) will follow if f is shown to be continuous at θ. Let $U = G \cap (-G)$ and let k be any positive integral upper bound for f on G. Then U is an open set containing θ and $|f(x)| \leq k$ for any x in U. Let n be any positive integer and set $V = U/nk$. Then V is an open set containing θ, and the additivity of f implies that $|f(x)| \leq 1/n$ for x in V. Thus f is continuous at θ. If

f is constant on G it must vanish on any translation W of G such that $\theta \in W$; it must then vanish at each x in E, for since W is open it contains x/n for some positive integer n, so that $0 = f(x/n) = f(x)/n$. If f is not constant on G then $f(G)$ is, by Lemma 1, a non-void set of reals. The case in which f is bounded below follows on applying the preceding to $-f$.

We come now to the very important "Separation Theorem", which has had a long and highly useful career. Given two sets P and Q in E and a functional f on E, we say *f essentially separates P and Q* if $\inf f(P) \geq \sup f(Q)$, and *f separates P and Q* if $\inf f(P) > \sup f(Q)$; if there exists an element of $\widehat{E}\setminus 0$ that (essentially) separates P and Q then P and Q are *(essentially) separate*. It is easy to verify that an additive functional (essentially) separates two sets P and Q if and only if it (essentially) separates $P-Q$ and θ, and a continuous one if and only if it (essentially) separates P^* and Q^*. From this it is easy to see that the following holds.

Lemma 3. *If in any one of the following pairs the two members are (essentially) separate then in every pair the two members are (essentially) separate: P and Q, $P - Q$ and θ, P^* and Q^*, $(P - Q)^*$ and θ^*, $(P - Q)^*$ and θ, $P - Q$ and θ^*, P and Q^*, P^* and Q.*

The Separation Theorem can now be stated as follows.

Theorem 3. *Suppose P and Q are sets in F such that $(P - Q)^*$ is a midpoint convex body. They (1) P and Q are essentially separate if and only if θ is not in the interior of $(P-Q)^*$, and (2) P and Q are separate if and only if θ is not in $(P - Q)^*$.*

From Lemma 3 we know that P and Q are (essentially) separate if and only if $(P - Q)^*$ and θ are; in view of this the proof reduces to showing that if A is any closed midpoint convex body then A and θ are essentially separate if and only if $\theta \notin A^\circ$ and are separate if and only if $\theta \notin A$. The proofs of the "only if" parts of the last statement are simple. If A and θ are separate obviously θ can't be be in A. And if A and θ are essentially separated by f then $\inf f(A) \geq f(\theta) = 0$ and $f(A^\circ)$ is by Lemma 1 an open set; since an open set of non-negative reals can't contain 0, clearly A° can't contain θ.

It remains to show that if $\theta \notin A$ then $\inf f(A) > 0$ holds for some f in $\widehat{E}\setminus 0$, and that if $\theta \notin A^\circ$ then $\inf f(A) \geq 0$ for some such f. In either case θ fails to be in A°; on choosing some element a of A° and letting $B = a - A$, B is a closed midpoint convex body whose interior contains θ but doesn't contain a; if moreover $\theta \notin A$ then $a \notin B$. Letting p_B be the Minkowski functional of B, we recall from Theorem 2 that $B = \{x\colon x \in E,\ p_B(x) \leq 1\}$ and $B^0 = \{x\colon x \in E,\ p_B(x) < 1\}$; hence $p_B(a) \geq 1$, and in case $\theta \notin A$ then $p_B(a) > 1$. Let $b = a/p_B(a)$, so that $p_B(b) = 1$, and let $F = \{\lambda b\colon \lambda \text{ real}\}$ and $g(\lambda b) = \lambda$ for each element λb of F. Since θ being in B° implies $p_B(\theta) = 0$, clearly $b \neq \theta$; thus g is well defined on the linear subspace F. And $g(\lambda b) = \lambda = \lambda \cdot p_B(b) = p_B(\lambda b)$ for $\lambda \geq 0$, and $g(\lambda b) = \lambda < 0 \leq p_B(\lambda b)$ for $\lambda < 0$. Thus $g \leq p_B$ on F. From this and the algebraic properties of g and p_B, it follows by the Hahn-Banach theorem [1] that there exists a real linear functional f on F that is an extension of g and is such that $f \leq p_B$ on E. Then $f(a) = g(a) = p_B(a) \neq 0$, and $\sup f(B) \leq \sup p_B(B) = 1$. Since $\sup f(B) \leq 1$ it follows from Lemma 2 that f is continuous; thus $f \in \widehat{E} \setminus 0$. Moreover, from the equality $f(A) = f(a - B) = f(a) - f(B) = p_B(a) - f(B)$ it is evident that $\inf f(A) = p_B(a) - \sup f(B) = p_B(a) - 1$. Hence $\inf f(A) \geq 0$; and if $\theta \notin A$ then $p_B(a) > 1$, as noted earlier in the proof, and so $\inf f(A) > 0$, completing the proof.

Corollary 3.1. *Let A and B be midpoint convex sets with A^0 and B non-void. Then these are equivalent: (1) there is some f in $E' \setminus 0$ such that $\inf f(A) \geq \sup f(B)$, (2) A and B are essentially separated, (3) A^0 and B are disjoint. Moreover, for any f such as described in (1) it follows that f is continuous, that $f(x) > \inf f(A^\circ) = \inf f(A^*) \geq \sup f(B^*)$ for any element x of A°, and that $\sup f(B^*) = \sup f(B^\circ) > f(y)$ in case B° is non-void and $y \in B^\circ$.*

Letting $P = A^\circ$ and $Q = B$, (3) is equivalent to θ not being in $P - Q$. But $P - Q$ is open and hence $P - Q = (P - Q)^\circ = ((P - Q)^*)^\circ$ by Corollary 2.1. Thus θ is not in $P - Q$ if and only if it is not in $((P - Q)^*)^\circ$. We can now apply the theorem and conclude that (3) is equivalent to A° and B being essentially separate. But by Lemma 3 the latter is equivalent to $(A^\circ)^*$ and B^* being essentially separate or, by Corollary 2.1, to A^* and B^* being separate in the same manner; this in turn, by Lemma 3 again, is equivalent to (2). Obviously (2) implies (1); and if (1) holds then f is bounded below on A° and so by Lemma 2 is continuous; thus (1) implies (2). The remaining equalities and inequalities are obvious consequences of the corollary to Lemma 1.

Corollary 3.2. *Let C be compact and B a midpoint convex set. Then C and B are separate if and only if there is an open convex set containing C that is disjoint with B.*

If C and B are separate and f is an element of $\widehat{E} \setminus 0$ such that $\inf f(C) > \sup f(B)$, the set $\{x\colon x \in E,\ f(x) > \sup f(B)\}$ is an open convex set containing G and disjoint with B. If on the other hand A is such a set then by Corollary 3.1 there is an f in $\widehat{E} \setminus 0$ such that $f(x) > \sup f(B^*)$ holds for each x in A and hence for each x in C. Since C is compact and f is continuous, $\inf f(C) = f(c)$ for some c in C, and therefore $\inf f(C) > \sup f(B^*)$, so that C and B are separate.

Recalling that a subset M of E is a *subspace* if $x+\lambda y \in M$ whenever x and y are in M and λ is arbitrary in S, and a *manifold* if it is a translate of a subspace, it is clear that on any manifold any linear functional is either constant or else assumes all values in S. From Corollary 3.1 we can now immediately infer

Corollary 3.3. *If A is a midpoint convex body and B is a non-void manifold then A° and B are disjoint if and only if there exists some f in $E'\setminus 0$ and some δ in S such that $f(x) \geq \delta$ for x in A and $f(y) = \delta$ whenever y is in B. Any such f is necessarily continuous, and $f(x) > \delta$ hold for each x in A°.*

A set M in E is a *hyperplane* (*continuous hyperplane*) if and only if there is some real number δ and some f in $E'\setminus 0$ $(\widehat{E}\setminus 0)$ such that $M = \{x\colon x \in E,\ f(x) = \delta\}$. A set N in E is *on one side of* a hyperplane M if for at least one such pair f and δ it is true that either $f(x) \geq \delta$ for every x in N or $f(x) \leq \delta$ for every x in N. The last corollary can now be rephrased as

Corollary 3.4. *When A is a midpoint convex body and B is a non-void manifold, A will lie on one side of some continuous hyperplane containing B if and only if A° and B are disjoint.*

Since for any f in $\widehat{E}\setminus 0$ and any real number δ the set $\{x\colon f(x) > \delta\}$ is a non-void open convex set, clearly $E\setminus H$ contains a convex body whenever H is a continuous hyperplane. This combined with Corollary 3.4 yields

Corollary 3.5. *A given manifold B lies in some continuous hyperplane if and only if $E\setminus B$ contains a convex body.*

Since any point in E is a manifold this in turn clearly provides the following.

Corollary 3.6. *$\widehat{E}\setminus 0$ is non-void if and only if there is a convex body in E that is a proper subset of E.*

If $A \subset E$ and H is a hyperplane in E then H is a *hyperplane of support of A* if A^* intersects H and A lies on one side of H, and is *a hyperplane of support of A at b* if $b \in A^* \cap H$ and A lies on one side of H. Consider a boundary point b of a midpoint convex body A; since $b \notin A^\circ$ it follows from Corollary 3.1, on setting $B = b$, that there is some f in $\widehat{E}\setminus 0$ such that $\inf f(A^\circ) = \inf f(A^*) \geq f(b)$, and since $b \in A^*$ it is also evident that the set $\{x\colon f(x) = f(b)\}$ is a continuous hyperplane of support of A at b. If conversely H is a continuous hyperplane of support of A at some point b in E then $b \in A^*$ and for some f in $\widehat{E}\setminus 0$ it is true that $\inf f(A^*) = f(b)$ and $H = \{x\colon f(x) = f(b)\}$. From the corollary to Lemma 1 it is obvious that b can't be in A°. Thus we have

Corollary 3.7. *Let A be a midpoint convex body and b a point in E. Then there is a continuous hyperplane of support of A at b if and only if $b \in A^* \setminus A^\circ$.*

Turning to a somewhat different situation suppose A and B are non-void disjoint midpoint convex sets such that $A \cup B = E$. (That such pairs exist in any non-trivial space was pointed out by Tukey [16].) Let $M = A^* \cap B^*$. Since $B^* = (E\setminus A)^* = E\setminus A^\circ$ and similarly $A^* = E\setminus B^\circ$, it follows that $M = A^* \setminus A^\circ = B^* \setminus B^\circ$ and that A° and B° are void together if and only if $M = E$. If either A° or B°, say A°, is not void then A and B satisfy the hypotheses of Corollary 3.1 so that there is some f in $\hat{E}\setminus 0$ such that $\inf f(A^*) \geq \sup f(B^*)$. Since $S = f(E) = f(A^* \cup B^*) = F(A^*) \cup f(B^*)$, clearly $\inf f(A^*) > \sup f(B^*)$ is impossible. Let n be the common value of $\inf f(A^*)$ and $\sup f(B^*)$. If $f(x) > n$ then manifestly x is not in B^* and so must be in A°; similarly, if $f(y) < n$ then $y \in B^0$. Since the converses of these two statements are also true by the corollary to Lemma 1, it follows that $A^\circ = \{x: f(x) > n\}$ and $B^\circ = \{y: f(y) < n\}$. Hence $\{x: f(x) \leq n\} = E\setminus A^\circ = B^*$ and $\{y: f(y) \geq n\} = A^*$. Then $M = \{x: f(x) = n\}$, and M is a continuous hyperplane. This has established

Corollary 3.8. *Let A and B be disjoint non-void midpoint convex sets such that $A \cup B = E$. Let $M = A^* \cap B^*$. Then either(1) $A^\circ \cup B^\circ$ is void and $M = E$ or else (2) both A° and B° are non-void and M is a continuous hyperplane having A on one side and B on the other.*

In the next three corollaries the topology of E is presumed to be *locally convex,* that is, each open set in E is the union of all the open convex sets it contains.

Corollary 3.9. *If P and Q are non-void sets in a locally convex space and $P - Q$ is midpoint convex, P and Q are separate if and only if θ is not in $(P - Q)^*$. Hence if $P - Q$ is closed and midpoint convex then P and Q are separate if and only if they are disjoint.*

Since the space is locally convex, θ fails to be in $(P - Q)^*$ if and only if there is an open convex set about θ that is disjoint with $P - Q$. But by Corollary 3.2 this is equivalent to $P - Q$ and θ being separate, and the latter by Lemma 3 is the same as P and Q being separate.

In case the topology of E is defined by a pseudo-metric ρ, the point θ fails to be in $(P - Q)^*$ if and only if $\inf \{\rho(p - q, \theta): p \in P, q \in Q\} \equiv$ distance $(P - Q, \theta) > 0$. If also ρ is invariant, i.e., $\rho(x,y) = \rho(x + z, y + z)$ for any x, y, z in E, then distance $(P - Q, \theta) =$ distance $(P,Q) = \inf \{\rho(p,q): p \in P, q \in Q$, and hence $\theta \not\in (P - Q)^*$ is equivalent to distance $(P,Q) > 0$. Thus we have

Corollary 3.10. *If the topology of E is locally convex and is defined by an invariant pseudo-metric and if $P - Q$ is midpoint convex then P and Q are separate if and only if distance (P, Q) is positive.*

This corollary applies to Banach spaces and in particular to n-dimensional Euclidean vector space.

It is easy to see that in a locally convex space $x + y$ is continuous in (x, y); for if G is open and $a + b \in G$ let W be an open convex set in G such that $a + b \in W$. Then $V \equiv W - a - b$ is open about θ and if $U = V/2$ we have $(a + U) + (b + U) = a + b + V \subset G$ where $a + U$ and $b + U$ are open about a and b respectively. Thus the space is a topological group under addition, and hence if P is a closed set and Q is compact the set $P - Q$ is closed [3]. The following is now an immediate consequence of Corollary 3.9.

Corollary 3.11. *If P and Q are midpoint convex sets in a locally convex space E and if P is closed and Q is compact then P and Q are separate if and only if they are disjoint.*

The space E is said to *have sufficiently many open convex sets* if given any two distinct points x and y in E there is an open convex set in E that contains one and only one of the points. Note that this implies that there are disjoint open convex sets G' and G'' about x and y respectively; for if H is open about y and $x \notin H$ let $U = (H - y) \cap (y - H)$ and $V = U/2$, and set $G' = x - V$ and $G'' = y + V$. This implies in particular that in any such space the topology is Hausdorff.

Corollary 3.12. *In a space E that has sufficiently many open convex sets suppose that A and B are such that $A - B$ is a compact midpoint convex set. Then A and B are separate if and only if they are disjoint.*

By Lemma 3 it is sufficient to consider A - B and θ, and by Corollary 3.2 it is enough to show that θ not being in A - B is equivalent to the existence of an open convex set containing θ but disjoint with $A - B$. If there is such a set clearly θ is not in $A - B$. Conversely, if θ is not in $A - B$ then for each point x in $A - B$ there are disjoint open convex sets G'_x and G''_x about x and θ respectively, and by the compactness of $A - B$ there are a finite number $x_1, \ldots, x_n$ such that $A \subset \bigcup_{i=1}^n G'_{x_i}$. The set $\bigcap_{k=1}^n G''_{x_i}$ is then clearly an open convex set about θ that is disjoint with $A - B$.

An obvious consequence of Corollary 3.12 is that any such space E *has sufficiently many continuous linear functionals*, that is, for any two distinct points x and y in E there is an element f in E such that $f(x) \neq f(y)$. Since the converse is obvious, the two properties are actually equivalent.

If it is assumed that $x + y$ is continuous in (x, y) then $x - y$ is also; hence if A and B are compact in such a space the set $A - B$ is the image under a continuous mapping of the compact set $A \times B$, so that $A - B$ is compact. There now results from Corollary 3.12 the following final separation result.

Corollary 3.13. *If A and B are two midpoint convex and compact sets in a space E that has sufficiently many open convex sets and in which $x + y$ is continuous in (x,y), the sets A and B are separate if and only if they are disjoint.*

The second main theorem is that of Minkowski, Price, and Krein and Milman, and asserts that in a suitably restricted space any compact convex set is determined by its "corner" points, in the sense that the closed convex cover of these points is the original set. The technical term for a corner point is "extreme point"; its definition is obtained as follows. A *face* of a convex set is any non-void convex subset F of A with the property that if $a \in A$, $b \in A$, and $\lambda a + (1 / \lambda) b \in F$ for some λ such that $0 < \lambda < 1$, then $a \in F$ and $b \in F$, i.e., if I is any interval $\{\lambda a + (1 - \lambda) b : 0 \leq \lambda \leq 1\}$ in A then F either is disjoint with I, or intersects I at a single end point, or contains I. A face of A consisting of a single point is called *an extreme point of A.* It is obvious that the intersection of any family of faces of A is again a face of A provided the intersection is non-void; and it is easy to see that if F is a face of A and G is a face of F then G is a face of A. A *functional face* of a convex set A is any subset F of A such that for some element f of $\widehat{E} \setminus 0$ it is true that $F = \{x: x \in A, f(x) = \sup f(A)\}$. Clearly every functional face of A is closed if A is closed, and non-void if A is compact: and every non-void functional face of A is a face of A.

Lemma 4. *If F is a compact closed face of a convex set A then F contains a compact minimal closed face M of A. If E has sufficiently many continuous linear functionals M is an extreme point of A.*

Since F is a closed face of A there is by the maximality principle of Hausdorff a family of closed faces of A that includes F as a member and is maximal with respect to the property that any finite number of members of the family have a non-void intersection. The intersection M of all members of the family is a closed convex subset of F and is non-void since F is compact; thus M is a closed face of A. From the maximality of the family, M clearly can contain no smaller closed face of A. If E has sufficiently many continuous linear functionals and M contains two distinct points, there is some f in $\widehat{E} \setminus 0$ that is not constant on M, and hence the set $N = \{x: x \in M, f(x) = \sup f(M)\}$ is a proper subset of M. But from some remarks above, N is a closed face of M and hence a closed face of A. Since M is minimal with respect to this property there is a contradiction, and M is therefore a single point.

Theorem 4. *A non-void compact convex set A in a space E having sufficiently many continuous linear functionals necessarily coincides with the closed convex cover B of its set of extreme points.*

By a remark above E is Hausdorff; hence A is closed and thus is a face of itself. From Lemma 4, A must contain at least one extreme point. Therefore B is non-void. Clearly $B \subset A$. If a is a point in A but not in B then since B is compact it follows from Corollary 3.12 that $\sup f(B) < f(a)$ for some f in $\widehat{E} \setminus 0$. Hence if $F = \{x: x \in A, f(x) = \sup f(A)\}$, then F is a closed face of A that is disjoint with B. But, by the above Lemma 4, F must contain an extreme point of A, and any such point is in B. This contradiction completes the proof.

Before beginning the final theorem a few additional definitions must be made. A *convex combination* of elements of a set M in any real linear space is any finite sum of the form $\sum_{i=1}^{n}\lambda_i x_i$ where each x_i is in M, each λ_i is non-negative, and $\sum_{i=1}^{n}\lambda_i = 1$. A set is convex if and only if it contains every convex combination of its elements. A real function ϕ on a convex set M is *convex* if $\phi(\sum_i\lambda_i x_i) \le \sum_i\lambda_i\phi(x_i)$ whenever $\sum_i\lambda_i x_i$ is any convex combination of elements of M, and *concave* if the reverse inequality always holds for any such combination. Recall that ϕ is *upper semi-continuous* if for any real number δ the set $\{x: \phi(x) \ge \delta\}$ is closed.

In these terms we can prove, using one of the above separation results, the fundamental theorem of game theory in the following form, which is a mild extension of Kneser's generalization [9] of Ville's theorem. The present form is essentially due to Kuhn [11, Th. 31], and the idea of the proof to Bohnenblust and Shapley [2].

Theorem 5. *Let X and Y be real linear spaces, A and B non-void convex subsets of X and Y respectively, and $\phi'(x,y)$ a function on $A \times B$ to the reals that is concave in x for each y in B and convex in y for each x in A. Suppose there is a topology in A such that A is compact and ϕ' is upper semi-continuous in x for each y in B. Then*

$$\sup_x \inf_y \phi'(x,y) = \inf_y \sup_x \phi'(x,y).$$

Let us first note two facts: (i) since A is compact and ϕ' is upper semi-continuous in x, the set $\{\bar{x}: \bar{x} \in A \text{ and } \phi'(\bar{x},y) = \sup \phi'(x,y)\}$ is non-void for each y; in particular, $\sup_x \phi'(x,y)$ is finite for each y; (ii) if $n = \inf_y \sup_x \phi'(x,y)$ then for any sets A and B and any real function ϕ' on $A \times B$ the inequality $\sup_x \inf_y \phi'(x,y) \le n$ holds.

From (i) clearly $n < \infty$, and from (ii) the desired inequality is obvious in case $n = -\infty$. We may thus suppose n to be finite. Denoting $\phi' - n$ by ϕ, so that ϕ inherits all the properties of ϕ' and $\inf_y \sup_x \phi(x,y) = 0$, we have to show that $\sup_x \inf_y \phi(x,y) \geq 0$.

To this end, for each y in B let $P_y = \{x\colon \phi(x,y) \geq 0\}$. If x_0 is any point in the intersection of all these sets then $\inf_y \phi(x\ ,y) \geq 0$; hence if this intersection is non-void clearly $\sup_x \inf_y \phi(x,y) \geq 0$ is true, and the proof would be finished. Now since A is compact and since for each y in B the function $\phi(x,y)$ is upper semi-continuous on A and hence has a non-negative supremum, each P_y is a non-void closed subset of A; hence, since A is compact, the intersection of all the P_y's will be non-void if and only if for any finite set $y_1, \ldots, y_n$ in B the set $\bigcap_{i=1}^n P_{y_i}$ is non-void.

We therefore suppose $\bigcap_{i=1}^n P_{y_i}$ to be void for such a set of y's, and endeavor to reach a contradiction. Define $\psi(x) = \{\phi(x,y\ ),\ \phi(x,y_2), \ldots, \phi(x,y_n)\}$ on A to E^n. Since $\bigcap_{i=1}^n P_{y_i}$ is void, clearly for each x in E the point $\psi(x)$ in E^n has at least one negative component; hence $P \cap \psi(A)$ is void, where P is the non-negative orthant in E^n. Let K be the convex cover of $\psi(A)$. If now it can be shown that the distance between K and P is positive then by a remark following Corollary 3.10 K and P are separate, and so there is a continuous non-zero linear functional γ defined on E^n such that $\sup\{\gamma(z)\colon z \in K\} < \inf\{\gamma(p)\colon p \in P\}$. Since $\theta \in P$ and $\lambda P \subset P$ for $\lambda \geq 0$, it is evident that $\inf\{\gamma(p)\colon p \in P\} = 0$; from this and the fact that e_i, the i^{th} unit vector in E^n, is in P it follows that $\gamma(e_i) \geq 0$ for each i. And γ being non-zero, clearly $\Sigma_i \gamma(e_i) > 0$. Let $\delta = \gamma/\Sigma\, \gamma(e_i)$. Then $\delta(z) = \Sigma_j \delta_j z_j$ for any element $z = \{z_1, \ldots, z_n\}$ in E^n, where $\delta_j = \delta(e_j) = \gamma(e_j)/\Sigma_i \gamma(e_i)$, $\delta_j \geq 0$ holds for each j and $\Sigma\delta_j = 1$. Clearly $\sup\{\delta(z)\colon z \in K\}$ is some negative number v, since this is true for γ. Hence $\delta(\psi(x)) \leq v$ for each x in A, i.e., $\Sigma_i \delta_i/\phi(x,y_i) \leq v$ for each i and each x in A; from the convexity of ϕ in y, the convexity of B, and the fact that $\Sigma_i \delta_i = 1$ where each $\delta_i \geq 0$, it follows that $\phi(x, \Sigma_i \delta_i y_i) \leq v$ for each x, and hence if $y_0 = \Sigma_i \delta_i y_i$ then $\sup_x \phi(x,y_0) < 0$ and hence $\inf_y \sup_x \phi < 0$, a contradiction as desired.

To complete the proof it remains to show that the distance from K to P is positive or, equivalently, that if $\rho(z)$ denotes the square of the distance from z to P for each z in E^n then $\inf\{\rho(z)\colon z \in K\} > 0$.

Note that $\rho(x)$ is, for any z, either 0 or the sum of the squares of the negative components of z. Now if $z \in K$, i.e., if z is a convex combination $\Sigma_{j=1}^{r}\lambda_j\psi(x_j)$ of elements of $\psi(A)$, then by the concavity of ϕ in x the i^{th} component of z is less than or equal to $\phi(\Sigma_{j=1}^{r}\lambda_j x_j, y_i)$, the i^{th} component of $\psi(\Sigma_j\lambda_j x_j)$, for each i. Since $\psi(\Sigma_j\lambda_j x_j)$ has at least one negative component, clearly then $\rho(z) \geq \rho(\psi(\Sigma_j\lambda_j x_j))$. From this it is obvious that $\inf\{\rho(z): z \in K\} \geq \inf\{\rho(z): z \in \psi(A)\}$. Thus it suffices to show that $\inf\{\rho(z): z \in \psi(A)\} = 0$ is impossible.

Suppose that for each positive integer n there is an x_n in A such that $\rho(\psi(x_n)) < 1/n$. Since A is compact there is some $\bar{x}$ in A at which the sequence $\{x_n\}$ clusters, i.e., every neighborhood of $\bar{x}$ contains infinitely many x_n's. Because $P \cap \psi(A)$ is void, $\phi(\bar{x}, y_k) < 0$ must hold for some k. Then, since $\phi(x, y_k)$ is upper semi-continuous in x and $\{x_n\}$ clusters at $\bar{x}$, we have $\lim_n \sup \phi(x_n, y_k) \leq \phi(x, y_k)$. Letting $\mu = \frac{1}{2}\phi(\bar{x}, y_k)$ there is some n_0 such that $\phi(x_n, y_k) \leq \mu < 0$ for all $n \geq n_0$. For such n clearly $\rho(\psi(x_n)) \geq \mu$; hence $\lim_n \inf \rho(\psi(x_n)) \geq \mu^2 > 0$ contrary to the fact that $\lim_n \rho(\psi(x_n)) = 0$. This ends the proof of Theorem 5.

Having completed the last of our theorems let us return to the "core" or "radial" topology for any real linear space that we defined in the introduction. The linear functionals that are continuous with respect to this topology are simply all the linear functionals. Hence for this topology there are sufficiently many continuous linear functionals (and therefore sufficiently many open, convex sets). Although the topology is not necessarily locally convex, there is an associated one, called the "convex core" or the "convex radial" topology, which is locally convex and has the same continuous linear functionals; in this topology a set is open if and only if it is the union of convex radially open sets. Moreover, $x + y$ is continuous in (x, y) although this may not be true for the original core topology.

In view of these remarks each theorem or corollary above yields a result stated in terms of the core or the convex core topology, concerning subsets of any real linear space. Theorem 3, for example, provides necessary and sufficient conditions, for certain pairs of sets P and Q, that P and Q can be (essentially) separated by a non-trivial linear functional. Corollary 3.1 becomes the following result of Dieudonné [5]: if A and B are two non-void convex sets in a real linear space E, and if A has at least one radial interior point, then B and the radial interior of A are disjoint if and only if there is a not-identically zero real linear functional f on E such that $\inf f(A) \geq \sup f(B)$; moreover, $f(x) > \sup f(B)$ for each x that is radially interior to A. Similarly Corollary 3.8 becomes a result of Klee's [8, (8.9)].

In conclusion certain examples should be mentioned. The first is due to Klee [8]. In any real linear space E having an infinite Hamel base with respect to the reals a proper subset P of E can be constructed that is convex, lies on one side of no hyperplane in E, has $-P \cup \theta$ as its complement, and is such that $-P \cup \theta$ is also convex. It is evident that P and $-P \cup \theta$ are complementary convex sets each everywhere dense in E with respect to every topology in E such that $x + y$ and ax are continuous in each variable. The second example is a variation on the first: any real linear space of more than one point has an infinite Hamel base with respect to the rationals, and by the same construction there will exist complementary midpoint convex sets that are everywhere dense with respect to every such topology. The last separation example will be only cited; it is due to Dieudonné [6], and consists of two bounded closed disjoint convex sets in the Banach space of absolutely convergent series that cannot be essentially separated. Finally Bourbaki [4] has an example showing the necessity of the compactness hypothesis in Theorem 4.

* * * * * * *

REFERENCES

[1] Banach, S.: *Théorie des Opérations Linéaires*, Warsaw, 1932.

[2] Bohnenblust, F. and Karlin, S.: *On a theorem of Ville*, Annals of Mathematics Studies 24 (1950), 155-160.

[3] Bourbaki, N.: *Topologie Générale*, Chapitres III,IV; Act. Sci. Ind. 916-1143, Paris, 1931.

[4] Bourbaki, N.: *Espaces Vectoriels Topologiques*, Chapitres I, II; Act. Sci. Ind. 1189, Paris, 1931.

[5] Dieudonné, J.: *Sur le théorème de Hahn-Banach*, La Revue Scientifique (Paris) 79 (1941), 642-643.

[6] Dieudonné, J.: *Sur le separation des ensembles convexes dans un espace de Banach*, La Revue Scientifique 81 (1943), 277-8.

[7] Kelley, J. L.: *Note on a theorem of Krein and Milman*, J. Osaka Inst. Sci. Tech. 3 (1951), 1-2

[8] Klee, V.L.: *Convex sets in linear spaces*, Duke Math.J. 18 (1951) 443-446.

[9] Kesner, H.: *Sur un théorème fondamental de la théorie des juex*, C.R. Acad. Sci. Paris 234 (1952), 2418-2420.

[10] Krein, M. and Milman, D.: *On extreme points of regular convex sets*, Studia Math. 9 (1940), 133-138.

[11] Kuhn, H. W.: *Lectures on the Theory of Games*, mimeographed Dept. of Mathematics, Princeton University, Princeton, N.J., 1953

[12] Minkowski, H.: *Gesammelte Abhandlungen*, Leipzig and Berlin 1911.

[13] Nikodym, O.M.: *Criteria for the continuity of linear functionals in real linear spaces*, Ohio J. of Sci. 52 (1952), 305-313.

[14] Price, G. B.: *On the extreme points of convex sets*, Duke Math. J. 3 (1937), 56-67.

[15] Stone, Marshall: *Convexity*, mimeographed, the Department of Mathematics, University of Chicago, Chicago, 1946; prepared by Harley Flanders.

[16] Tukey, J. W.: *Some notes on the separation of convex sets*, Portugaliae Mathematica 3 (1942), 95-102.

Tulane University

A CERTAIN BRIDGE TOURNAMENT SEATING PROBLEM AND LATIN SQUARES

James A. Ward

The following problem appeared in the American Mathematical Monthly in 1936[1]:

"How may eight married couples play a seven round bridge tournament, if each man plays one round with each lady except his wife, and everybody plays against everybody else except his or her spouse? How many solutions exist? Can this be generalized for 4 couples playing $4n - 1$ rounds? Does a solution exist for $4n + 2$ couples playing $4n + 1$ rounds?"

Bucker[2] gave a solution for eight couples and referred to the four-couple solution of Dudeney.[3]

The purpose of this paper is to show that any solution of the bridge-tournament seating problem for c couples determines three c by c latin squares,* orthogonal in pairs; to point out that no solution exist for $c = 6$ (a 3-table tournament); and to give the solution for $c = 2^k$ $(k > 1)$. Since the solution utilizes a field of characteristic 2. the author seriously doubts the existance of a sclution if $c \neq 2^k$.

Theorem 1. If this bridge tournament seating problem has a solution for c couples, there exist 3 c by c latin squares, orthogonal in pairs.

Let the c men be denoted by $m_1, m_2, \ldots, m_c$, and their respective wives by $w_1, w_2, \ldots, w_c$. In the r-th round of the seating arrangement denote the man opponent of m_i by m_{i_r} and the lady partner and opponent of m_i by w'_{i_r} and w_{i_r}, respectively.

Since each man plays against each man exactly once, the m_{i_r} $(r = 1, \ldots, c - 1)$ are all the men except m_i. Hence the set $m_i, m_{i_1}, \ldots, m_{i_{c-1}}$ is a permutation of $m_1, \ldots, m_c$ for $i = 1, 2, \ldots, c$. Furthermore $m_{i_r} = m_{j_r}$ would imply that in the r-th round m_i and m_j have the same man opponent, which is impossible. Therefore for every r the set $m_{1_r}, m_{2_r}, \ldots, m_{c_r}$ is a permutation of $m_1, \ldots, m_c$. Hence we see that

---

A latin square is an n by n array in which each of the numbers from 1 to n appears exactly once in each row and each column. For illustrations see the three4 by 4 latin squares just after Theorem 2 and the three 8 by 8 latin squares near the end of the paper.

Two n by n latin square (a_{ij}) and (b_{ij}) are onthogonal if the n^2 order pairs (a_{ij}, b_{ij}) are distinct. Each pair of latin squares in the examples given above is onthogonal.

$$A = \begin{bmatrix} m_1 & m_2 & \cdots & m_c \\ m_{1_1} & m_{2_1} & \cdots & m_{c_1} \\ \cdot & \cdot & \cdots & \cdot \\ m_{1_{c-1}} & m_{2_{c-1}} & \cdots & m_{c_{c-1}} \end{bmatrix}$$

is a c by c latin square.

Since m_i plays against each lady except his wife w_{i_r} $(r=1, 2, \ldots, c-1)$ are all the ladies except the wife of m_i. Therefore w_i, w_i, ..., $w_{i_{c-1}}$ is a permutation $w_1, \ldots, w_c$. Since no lady can play against two men simultaneously the $w_{1_r}, \ldots, w_{c_r}$ are distinct for each r and

$$B = \begin{bmatrix} w_1 & w_2 & \cdots & w_c \\ w_{1_1} & w_{2_1} & \cdots & w_{c_1} \\ \cdot & \cdot & \cdots & \cdot \\ w_{1_{c-1}} & w_{2_{c-1}} & \cdots & w_{c_{c-1}} \end{bmatrix}$$

is a c by c latin square. Likewise w_i, w'_{i_1}, ..., $w'_{i_{c-1}}$ are distinct and w'_{1_r}, ..., w'_{c_r} are distinct so that

$$C = \begin{bmatrix} w_1 & w_2 & \cdots & w_c \\ w'_{1_1} & w'_{2_1} & \cdots & w'_{c_1} \\ \cdot & \cdot & \cdots & \cdot \\ w'_{1_{c-1}} & w'_{2_{c-1}} & \cdots & w'_{c_{c-1}} \end{bmatrix}$$

is a c by c latin square.

The pair (m_{i_r}, w_{i_r}) are opponents against m_i in the r-th round. If this pair is the same pair as (m_{j_s}, w_{j_s}) then a particular man and lady are partners twice, contrary to hypothesis. Nor can (m_{i_r}, w_{i_r}) equal (m_u, w_u), for no man plays with his wife. Hence A and B are orthogonal. Similar arguments show that A and C are orthogonal and also B and C, and the theorem is proved.

Corollary. This bridge tournament seating arrangement is impossible for a 3-table tournament.

This would require 6 couples and by theorem 1 would require 3 6 by 6 orthogonal latin squares which Tarry[4] showed to be impossible.

We can determine this bridge tournament seating arrangement for $c=2^k$ couples (2^{k-1} tables) is the following way:

Denote the elements of the Galois field of order $c=2^k$ by $a_1=0$, $a_2=1, a_3, \ldots, a_c$. Then

(1) $$L_u = \begin{bmatrix} a_1a_u + a_1 & a_1a_u + a_2 & \cdots & a_1a_u + a_c \\ a_2a_u + a_1 & a_2a_u + a_2 & \cdots & a_2a_u + a_c \\ \cdots & \cdots & \cdots & \cdots \\ a_ca_u + a_1 & a_ca_u + a_2 & \cdots & a_ca_u + a_c \end{bmatrix}$$

is a latin square for $u = 2, 3, \ldots, c$ and these squares are orthogonal in pairs[5].

Let us set up the correspondences:

(2) $a_{r+1}a_\alpha + a_i \longleftrightarrow m_{i_r}$, $a_{r+1}a_\beta + a_i \longleftrightarrow w_{i_r}$, $a_{r+1}a_\gamma + a_i \longleftrightarrow w'_{i_r}$ in which α, β, and γ are to be determined. Under this correspondence the i-th columns (with each top element deleted) of L_α, L_β and L_γ will determine a seating arrangement (the men opponents, lady opponents, and lady partners, respectively) for each m_i.

(3)

round	man opponent	lady opponent	lady partner
1	m_{i_1}	w_{i_1}	w'_{i_1}
. . .	. . .	. . .	. . .
$c-1$	$m_{i_{c-1}}$	$w_{i_{c-1}}$	$w'_{i_{c-1}}$

We will now show that these are the required seating arrangements. Since the elements of the i-th column of L_α are distinct, $m_{i_1}, \ldots, m_{i_{c-1}}$ are distinct and none of them equals m_i, so that m_i has each man as his opponent. Since the elements of the i-th column of L_β are distinct and none of them is w_i, so that m_i plays against each lady except his wife. In the same manner we show that m_i plays with each lady except his wife as a partner. By the method used in the proof of theorem 1 it can be shown that the orthogonality of L_α, L_β, and L_γ will prevent people playing as partners more than once or as opponents more than once.

Finally we must show that the set (3) is consistent, that is: if in the r-th round, m_i and m_j are opponents, the lady opponent and partner of m_i must be the lady partner and opponent, respectively, of m_j. If in the r-th round m_j is the opponent of m_i, and conversely, we have by (2)

$$a_j = a_{r+1}a_\alpha + a_i, \quad a_i = a_{r+1}a_\alpha + a_j$$

Hence

(4) $$a_{r+1}a_\alpha = a_j - a_i = -(a_j - a_i)$$

which holds because a field of order 2^k has characteristic 2. For their lady partners and opponents we must have

$$w_{i_r} = w'_{j_r}, \quad w'_{i_r} = w_{j_r}$$

which holds by (2) if and only if

$$a_{r+1}a_\beta + a_i = a_{r+1}a_\gamma + a_j, \quad a_{r+1}a_\gamma + a_i = a_{r+1}a_\beta + a_j$$

From (4) either of these gives

$$a_{r+1}a_\beta - a_{r+1}a_\gamma = a_{r+1}a_\alpha$$

Since $a_{r+1} \neq 0$ for $r = 1, \ldots, c-1$ we have

$$a_\beta = a_\alpha + a_\gamma$$

which is independent of r, and hence holds for every round.

From (1) we must have $a_\alpha a_\beta a_\gamma \neq 0$. If $a_\alpha = a_\beta$, $a_\gamma = 0$. Hence by (5) L_α, L_β and L_γ can be chosen in $(2^k-1)(2^k-2)$ ways. Thus we have proved

Theorem 2. This bridge tournament seating arrangement for 2^k couples may be accomplished in $(2^k-1)(2^k-2)$ ways from any three distinct latin squares (1) such that $a_\alpha = a_\beta + a_\gamma$.

Hence there are 6 ways of constructing a 4-couple (2-table) bridge tournament. The conditions of theorem 2 are satisfied if we take L_α, L_β, and L_γ to be respectively:

$$\begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ a_2 & a_1 & a_4 & a_3 \\ a_3 & a_4 & a_1 & a_2 \\ a_4 & a_3 & a_2 & a_1 \end{pmatrix}, \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ a_3 & a_4 & a_1 & a_2 \\ a_4 & a_3 & a_2 & a_1 \\ a_2 & a_1 & a_4 & a_3 \end{pmatrix}, \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ a_4 & a_3 & a_2 & a_1 \\ a_2 & a_1 & a_4 & a_3 \\ a_3 & a_4 & a_1 & a_2 \end{pmatrix}.$$

Following the method used in the proof of theorem 2 we find the seating arrangement for each m_i by taking the i-th columns of L_α, L_β, and L_γ (with top element deleted) as the elements corresponding to the men opponents, lady opponents, and lady partners, respectively, of m_i. The arrangement follows:

For m_1

Round	Man opponent	Lady opponent	Lady partner
1	m_2	w_3	w'_4
2	m_3	w_4	w'_2
3	m_4	w_2	w'_3

For m_2

Round	Man opponent	Lady opponent	Lady partner
1	m_1	w_4	w'_3
2	m_4	w_3	w'_1
3	m_3	w_1	w'_4

For m_3

Round	Man opponent	Lady opponent	Lady partner
1	m_4	w_1	w'_2
2	m_1	w_2	w'_4
3	m_2	w_4	w'_1

For m_4

Round	Man opponent	Lady opponent	Lady partner
1	m_3	w_2	w'_1
2	m_2	w_1	w'_3
3	m_1	w_3	w'_2

For an 8-couple (4 table) bridge tournament we can choose the latin squares in 42 ways. Three latin squares which fulfill the conditions of theorem 2 are (only the subscripts are given):

1	2	3	4	5	6	7	8
2	1	4	3	6	5	8	7
3	4	1	2	7	8	5	6
4	3	2	1	8	7	6	5
5	6	7	8	1	2	3	4
6	5	8	7	2	1	4	3
7	8	5	6	3	4	1	2
8	7	6	5	4	3	2	1

1	2	3	4	5	6	7	8
5	6	7	8	1	2	3	4
2	1	4	3	6	5	8	7
6	5	8	7	2	1	4	3
7	8	5	6	3	4	1	2
3	4	1	2	7	8	5	6
8	7	6	5	4	3	2	1
4	3	2	1	8	7	6	5

1	2	3	4	5	6	7	8
6	5	8	7	2	1	4	3
4	3	2	1	8	7	6	5
7	8	5	6	3	4	1	2
3	4	1	2	7	8	5	6
8	7	6	5	4	3	2	1
2	1	4	3	6	5	8	7
5	6	7	8	1	2	3	4

The seating arrangement for each m_i is found by taking the i-th column (with the top element deleted) of each of the three latin squares above. The three columns will denote the men opponents, lady opponent, and lady partners, respectivelu of m_i. For brevity we give only the subscripts.

Round	m_1	m_2	m_3	m_4	m_5	m_6	m_7	m_8
1	256	165	478	387	612	521	834	743
2	324	413	142	231	768	857	586	675
3	467	358	285	176	823	714	641	532
4	573	684	751	862	137	248	315	426
5	638	547	816	725	274	183	452	361
6	782	871	564	653	346	435	128	217
7	845	736	627	518	481	372	263	154

* * * * * * *

FOOTNOTES

1. G. A. Whittemore, American Mathematical Monthly, Vol. 43 (1936) p. 242
2. W. E. Bucker, American Mathematical Monthl u, Vol. 45 (1938) p. 479
3. H. E. Dudeney, Amusements in Mathematics, p. 203
4. G. Tarry, Le Probleme des 36 officiers, Association Francaise pour L'Avancement des Sciences, Comptes Rendus, Vol. 29 (1901) pp. 170-203.
5. H. B. Mann, Analysis and Design of Experiments, Dover Publications, New York, 1949, Chapter 8, pp. 87-106

University of Kentucky

LONG - SHORT LINES

Glenn James

Forward. It is not what mathematicians say but the way they say it that makes mathematics almost un-understandable to the layman, the student and the mathematician who is not an expert in the subject matter at hand. Thus the highly concentrated, ideosyncratic language of mathematics is a handicap as well as an essential aid in the development of the subject. This handicap can be removed by translating mathematical treatments into plain English* or, in the case of students, introducing them to mathematics in plain English. The following paper attempts to present the principle of a sequence(in this instance a sequence of sets of curves) approaching a limit which differs radically from the sets of curves, in understandable language.

The length of the strip of carpet required to cover a set of stairsteps rising from a fixed point on one floor to a fixed point on a higher floor is the same regardless of how many steps there are. It is equal to the horizontal distance plus the vertical distance covered by the stairway, allowing perhaps a small difference due to turning corners.

A few years ago a professor from another department came into my office awed by having discovered (for himself) the above fact, awed not so much by the above fact as by the situation to which it leads. He had drawn triangles such as those below (in Fig. I).

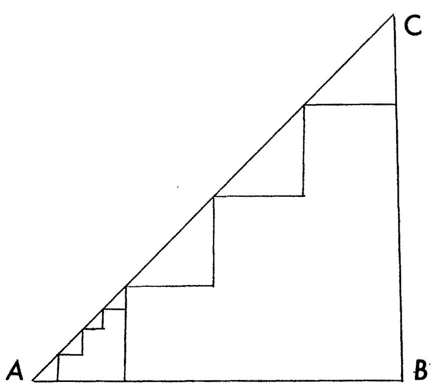


Fig. I

* Current publications are doing considerable along this line. There comes to my mind just now the publications of the Galois Institute, some articles in the Scientific American and the standing request of the Canadian Journal of Mathematics that its authors provide summarizing introductions to their articles, which can "be understood by the non-expert".

Consider the four middle size triangles. It is quite evident that the distance $AB+BC$ is equal to the sum of the legs of the small triangles and that we could draw four smaller triangles in each of the small ones thus getting sixteen small triangles the sum of whose legs is equal to the sum of the legs of the first set of triangles hence equal to $AB + BC$. Now an extraordinary situation begins to appear. You can keep replacing your sets of small triangles by sets of smaller ones until you can no longer distinguish your row of small triangles from the straight line AC. Yet the distance along the jagged line from A to C will still be equal to $AB + BC$.

Now one wonders if a similar situation would exist if he replaced triangle ABC by a semicircle with diameter AC and the small triangles by semicircles as shown in Fig. II.

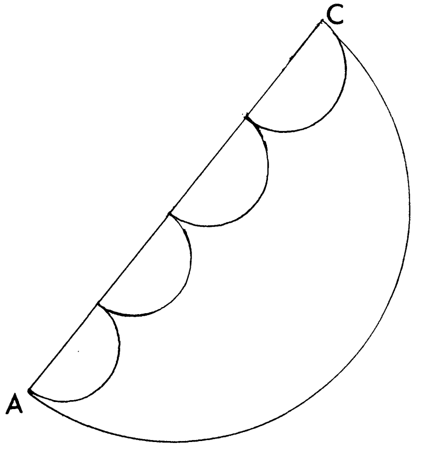


Fig. II

A similar situation does exist for the distance along the large semicircle is equal to the distance along the row of small semicircles. We prove this by recalling that the length of a semicircle is equal to the diameter times $\pi/2$. Hence the length of the large semicircle is AC times $\pi/2$ while the distance along the small semicircles is the sum of their diameters each multiplied by $\pi/2$ but the sum of their diameters is exactly AC hence the distance along the small semicircles is also AC times $\pi/2$. Now if you make the diameters of the small circles sufficiently small you can no longer distinguish the row of little circles from the straight line AC. Yet the distance along the little circle path is not the same as along CD. It is as we have shown the same as the distance along the large semicircle.

Another such situation arises when one compares the curve made by a point on a large wheel, as the wheel rolls along a straight path, with the curve made by a point on a smaller wheel, as it too rolls along the path. These curves are pictured in a reduced drawing in Fig. III.

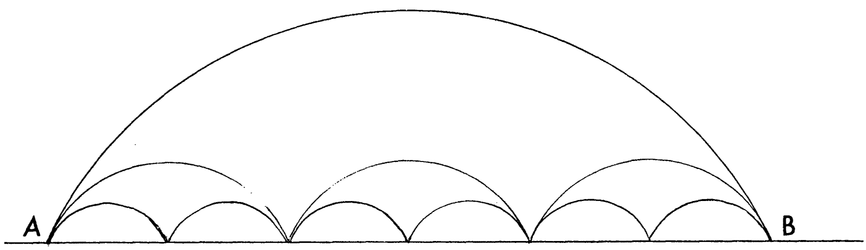


Fig. III

It can be shown by the use of calculus that the curve from A to B is the same length whether you travel along that made by the large wheel or along that made by the small one. If you would use a large wheel with diameter one hundred feet, say, and for the small one a fine needle, the distances along the two paths would still be equal. Again, if the small wheel be so small that you cannot distinguish its path from the straight line AB, the distance along the pinkley line would still be equal to that along the curve made by the wheel whose diameter is one hundred feet.

In the three cases just considered the rows of small figures, the triangles, the semicircles and the cycloids (for that is what we call the curves in Fig. III) *approach the straight lines which they lie along as they become smaller and their numbers become correspondingly larger; but their lengths do not approach the lengths of these straight lines.*

For convenience let us call these apparent straight lines, of unusual length, *pseudo-lines*. By "apparent straight lines" we mean our sets of small figures after they have become, individually, so small (and their numbers so great) that one cannot distinguish them from straight lines.

One now begins to wonder what kinds of curves one can have such that sets (families) of them will approach a pseudo-line, and what lengths pseudo-lines can have.

Let us consider one little typical arch in some family of curves such as in Fig. IV.

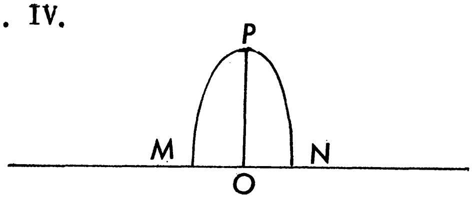


Fig. IV.

Suppose OP is greater than MN. Then MPN is greater than $2MN$. If this be true of all the little arches in a family of curves which defines a pseudo-line, then this pseudo-line is more than twice as long as the proper line which serves as a base. Again, if OP be ten times as long as MN, then the corresponding pseudo-line would be at least twenty times as long as the base line, and if OP be a million times as long as MN the corresponding pseudo-line would be more than two million times as long as the base line. Incidently, it would seem to be dangerous to say that an apparent straight line is the shortest distance between its end points, for its length depends upon how it is generated.

Some General Observations. If we think of our family of curves as a cross section of an undulating surface like a corduroyed road, one can easily see how we could have a *small* area which considered as the product of a pseudo-line by a constant width would have an arbitraily large, apparently plane, area.

Obviously in the sets of small arches, we have considered, not all would need to be similar nor indeed of the same height. It would suffice if they all had the same ratio between their OP's and MN's (see Fig. IV) and the heights of the highest of them approached zero.

Moreover, instead of a line for base we might have some fixed curve. For example, we might consider the arches developed by fixed points on small circles rolling around the outside (or inside) of a large fixed circle thus developing what are known as epicycloids and hypocycloids. If the lengths of the rolling circles are taken smaller and smaller the families of arches which they develop approach the fixed circle as a limit but the lengths around these sets of arches does not approach the perimeter of the fixed circle. Thus they develope what we may call pseudo-circles. This concept can obviously be extended to an area of any curve*, which may be a section of an arbitrarly large pseudo-area that appears to be a small actual area. This is a matter of considerable practical importance, for instance when we are concerned with heat flow through a surface.

*For an analytic treatment of the above concept from a slightly different viewpoint, readers who have the prerequisites for the calculus of variations might like to read: Tohoku Mathematical Journal Vol.39, pp. 322-326 and Tonelli, Calcolo de Variszzoni Vol. I, pp.95-105.

CURRENT PAPERS AND BOOKS

Edited by

H.V. Craig

This department will present comments on papers previously published in the MATHEMATICS MAGAZINE, lists of new books, and book reviews.

In order that errors may be corrected, results extended, and interesting aspects further illuminated, comments on published papers in all departments are invited.

Communications intended for this department should be sent in duplicate to H.V. Craig, Department of Applied Mathematics, University of Texas, Austin 12, Texas.

Comment on L. E. Diamond's "New Meanings for Old Symbols"*

H. W. Becker

The table of "Colors as Number Symbols" (p. 212) is sometimes supplemented by 2 colors, gold and silver, which appear only in the 3d and 4th bands, with entirely different meanings in the 2 bands. In the 3d band, gold and silver mean multiplication by 10% and 1% respectively. Thus brown-black-gold means (10)(.1) = 1 ohm. In 33 years experience in electronics, it so happened I'd never seen precisely such color-coded resistors, until finding them on the lab experiment resistor boards used by all students at R.E.I. (Why could not black-brown-black also be used, to indicate 1 ohm? Well, it just isn't done, I've never seen black in the 1st band.) This enables color-coding from 10 down into the range of fractional ohms.

In the 4th band, as Mr. Diamond knows, gold and silver mean a maximum tolerance of 5% and 10% resp., since resistors are difficult to mass-produce with high precision (attained by filing notches where necessary, as in multimeters). Absence of the 4th band is supposed to denote a maximum tolerance of 20%; which is a good joke, because tolerances of 50% are frequently encountered. Many circuits aren't critical anyway, of these wide tolerances. But TV sweep circuits are: if your picture rolls or tears, maybe some resistor has changed its value too much! Tolerance color-coding of course corresponds to rounding off, or probable error, in ordinary mathematics and science. In this Electronic Computer Age, any mathematician might have to learn the "lingo" of these "gismoes".

---

* MATHEMATICS MAGAZINE, 29 (1956) 290-16. See p. 212.
Also, A. Marcus, *RADIO SERVICING* (Prentice-Hall, NYC, 1954) p.99.

Radio Engineering Institute, Omaha, Nebraska.

The Trapezoid of Two Crossed Ladders

C. D. Smith

In MATHEMATICS MAGAZINE, Vol. 29, No. 3, we have a discussion by H. A. Arnold where z is calculated in terms of the height of the crossing point as unit. To obtain any required length in customary units such as feet, we may let the height of the crossing point be represented by c. The three relations follow:

$$(1)\quad c(y + x) = xy; \qquad (2)\quad z^2 + x^2 = a^2; \qquad (3)\quad z^2 + y^2 = b^2.$$

It should be of interest to extend the discussion to cases where three of the six lines are given and the remaining three lines are required. The case discussed by Arnold, where $a, b, c,$ are known may be reduced without further transformations as follows. From (2) and (3) $y^2 - x^2 = b^2 - a^2$. This result combined with (1) gives either x or y in an equation of fourth degree. Values of x, y, and z follow.

There are 20 combinations of 6 lines taken 3 at a time. The solution referred to leaves 19 cases for consideration. If we know x, z, and a, it follows that y may very and a is the locus of the cross point. There is no unique solution. If y, z, and b are known, the value of x may vary and the cross point has the locus b. If x, y, and c are known, we have the interesting case where c is the same for every z. In fact, the locus of the cross point is a straight line parallel to z. This must be true since equation (1) gives the relation which is independent of z in (2) and (3).

We now have 16 cases remaining where we expect unique solutions. In each of these cases, the solution follows directly from (1), (2), and (3). In conclusion we can say that the problem has 20 cases with 3 parts given, of which three cases are indeterminate and only one case, with a, b, c, given, which leads to an equation of fourth degree.

Differential Equations. By Harry W. Reddick and Donald E. Kibbey John Wiley & Sons, New York, 1956, 304 pages. $4.50

The third edition of *Differential Equations* was published in March. More than half rewritten, the book brings up to date an already well-known text.

Differential equations deals with methods of solving ordinary differential equations and with problems in applied mathematics involving their use. The third edition contains a new chapter on partial differential equations and a section on the adjoint equation, the latter appearing in the revised discussion of the linear equation of second order. The authors have also effected many additions, changes, and rearrangements involving the other chapters: preliminary ideas, differential equations of first order, linear equations with constant coefficients, some special higher order equations, simultaneous equations, and series solutions.

Richard Cook

Spheroidal Wave Functions. By J.A. Stratton, P.M. Morse, L.J. Chu, J.D.C. Little, and F.J. Corbató. John Wiley & Sons and The Technology Press of the Massachusetts Institute of Technology. 1956, 613 pages $12.50.

This book is an exhaustive compilation of tables of separation constants and coefficients, It was published in January under the joint imprints of the above publishers and is the work of five specialists.

Dealing with spheroidal wave functions appropriate for prolate or oblate spheroidal boundaries, the volume defines certain standard forms of their solution that are of greatest practical use. The book also displays a collection of formulas giving the important mathematical properties of these functions, and provides a set of tables from which values of the solutions can be obtained in the more interesting range of variables. These tables contain the series coefficients, together with some of the separation constants and allied coefficients.

Superseding *Elliptic Cylinder and Spheroidal Wave Functions,* the present work affords greater access to calculations that can be put to work in the fields of applied physics, acoustics, and radar. The tables permit the handling of wave problems in spheroidal coordinates with approximately the same degree of facility possible for rectangular, circular, cylindrical, and spherical coordinates. All the numbers contained here were determined by the high speed electronic digital computer Whirlwind I at M. I. T. Computed, tabulated, and printed automatically, the collection takes on a unique degree of accuracy otherwise unobtainable. The tables are accompanied by considerable theoretical material.

Richard Cook

Table of the Descending Exponential, $x = 2.5$ *to* $x = 10$, National Bureau of Standards Applied Mathematics Series 46, 76 pages, 50¢. (Order from Government Printing Office, Washington 25, D.C.).

The descending or negative exponential e^{-x} is tabulated in this publication from $x = 2.5$ to $x = 10$ at intervals of .001 to 20 decimal places. The table provides greater accuracy than has heretofore been available in this range of the argument and hence facilitates interpolation.

Many laws of nature in the realm of classical and modern physics, including classical mechanics, kinetic theory of matter, statistical mechanics, electricity, magnetism, and quantum mechanics, are expressed in the form of ordinary or partial differential equations, the solutions of which involve exponential functions. This also applies in certain branches of physical chemistry, such as chemical kinetics. Exponential functions are widely used in the theory of probability and statistics.

The table in this volume is a supplement to the tables of the exponential function e^x that were prepared in 1939 by the New York Mathematical Table Project under the Bureau's scientific sponsorship. They have twice been reprinted - in 1947 as Mathematical Table MT2, second edition, and in 1951 as Applied Mathematics Series 14. In the earlier volume the descending exponential is given from $x=0$ to $x=2.5$ at intervals of .0001 to 18 decimal places and for $x=1$ to $x=100$ at intervals of 1 to 19 significant figures.

(NOTE: Foreign remittances must be in U. S. exchange and should include an additional one-third of the publication price to cover mailing costs).

The Mathescope. By C. Stanley Ogilvy. Oxford University Press 1956, illustrated.

Mr. Ogilvy acquaints the layman with the intellectually stimulating aspects of various branches of mathematics. Combining an introduction to the subject with actual problems and examples, he gives the reader a better understanding of the theorectical material. Instead of going over any of the old and tiresome school-room topics, the author explores some of the less familiar by-paths from which may be obtained interesting glimpses of the kind of things with which mathematicians concern themselves. Quick to dispel the impression that mathematics is no longer a fruitful area of investigation, the book presents mathematics as a vital subject with constantly broadening frontiers. The more serious discussions are interlarded with many amusing illustrations.

Book News-Oxford Press

The French Bibliographical Digest. Edited by the Cultural Division of the French Embassy, 972 Fifth Avenue, New York 21.

This sixth issue of the second series of the *French Bibliographical Digest* is devoted to Mathematics. Although originally it had been planned to cover in a single volume French contributions to the whole field of Mathematics, the abundance of material has prompted us to prepare two separate issues. The present one deals with Pure Mathematics. Part II which is to be published shortly will deal with Applied Mathematics.

The French Bibliographical Digest is intended primarily to make French scientific work better known in the United States. Libraries, university departments and scientists will, upon request, be placed on our mailing list and receive the publications without charge. To avoid indiscriminate distribution, our correspondents are advised to specify the particular field of their interest.

Pierre Donzelot
Permanent Representative of
French Universities in the United States

An Introduction to Mathematics: A Historical Development, Revised Edition. By Lee Emerson Boyer. Henry Holt and Company.

This book is the revised edition of the book formerly titled *An Introduction To Mathematics For Teachers* and the alternate edition titled *Mathematics, A Historical Devdlopment*. In the main, the body of the test remains unchanged. Most of the problems and exercises have been changed and the number of problems increased. In addition to the greater number of problems in the main text a large set of supplementary problems and exercises is found in the Appendix. New sections have been added on installment buying and bond purchasing. The supplementary section on algebra contains new illustrations of the uses of this branch of mathematics. The Appendix also contains a set of tables which eliminates the need for outside references as far as mathematics table are concerned.

For those readers who are not acquainted with the first edition, the book is well suited to provide content mathematics for prospective arithmetic teachers or to supply a terminal mathematics program for general education purposes. Although the text is written in a simple style it introduces, and even stresses, the philosophy and fundamental concepts of mathematics. It stresses mathematical meanings throughout by the spirit of its writing and the practical character of the problems.

Lee E. Boyer.

Numerical Analysis. By Zdenek Kopal. John Wiley & Sons, 1956. 440 Fourth Ave. New York 16, N.Y. 556 pages, $12.00.

An introduction to the numerical analysis of the functions of a single real variable, the new book emphasizes the application of numerical techniques to problems of infinitesimal calculus.

An introductory chapter gives the present status and achievements of numerical analysis and places them in historical perspective. In Chapters Two through Six, Dr. Kopal supplies those elements of the theory and practice of interpolation (based on polynomial approximation) which are necessary for numerical integration and differentiation, for the solution of ordinary differential equations of any order or degree for initial-value problems, and of linear differential equations of any order for boundary-value problems. Chapter Seven provides a

Richard Cook

TEACHING OF MATHEMATICS

Edited by

Joseph Seidlin and C. N. Shuster

This department is devoted to the teaching of mathematics. Thus articles on methodology, exposition, curriculum, tests and measurements, and any other topic related to teaching, are invited. Papers on any subject in which you, *as a teacher*, are interested, or questions which you would like others to discuss, should be sent to Joseph Seidlin, Alfred University, Aflred, New York.

Dear Mr. Seidlin:

For some time I have been somewhat annoyed with the treatment in several books of the Mean Value Theorems and extensions. I have therefore written the enclosed article in the belief that other teachers may be interested in it. I shall be pleased if you find it suitable for publication in your magazine.

Sincerely yours,

Murray R. Spiegel.

MEAN VALUE THEOREMS AND TAYLOR SERIES

M. R. Spiegel

In all of the textbooks on elementary and advanced calculus with which the author is acquainted, the various mean-value theorems and Taylor series with a remainder are arrived at by setting up a function judiciously and then applying Rolle's Theorem. In many cases the student justifiably may get the feeling that this "suitable" function is pulled out of the proverbial hat. The purpose of this article is to present a basis for arriving at these "suitable" functions. The method employed does not seem to be well-known, but appears to be valuable from a pedagogical as well as theoretical viewpoint.

We begin by stating the theorem due to Rolle which will be referred to throughout the paper.

Rolle's Theorem: Let $F(x)$ be defined, single-valued and continuous in the interval $a \leq x \leq b$ and be such that $F(a) = F(b) = 0$. Furthermore let $F(x)$ have a derivative (finite or infinite) in the open interval $a < x < b$. Then there is at least one number ξ such that $a < \xi < b$ and for which $F'(\xi) = 0$.

This theorem is adequately proved in many texts although they seem to avoid unnecessarily the case where $F(x)$ has an infinite derivative.

Let us now consider a function $f(x)$ defined, single-valued and continuous in $a \leq x \leq b$ and having a derivative in $a < x < b$. We seek to approximate $f(x)$ in the interval $a \leq x \leq b$ by a linear function $A + Bx$ where A and B are constants which we shall determine. Let us consider the difference between $f(x)$ and the linear function $A + Bx$ and write

$$F(x) \equiv f(x) - (A + Bx) \qquad a \leq x \leq b. \tag{1}$$

We now choose the constants A and B so that $F(a) = F(b) = 0$. That is

$$F(a) \equiv f(a) - (A + Ba) = 0 \tag{2}$$

$$F(b) \equiv f(b) - (A + Bb) = 0 \tag{3}$$

Since $F(x)$ now satisfies the conditions of Rolle's Theorem, it follows that

$$F'(\xi) \equiv f'(\xi) - B = 0 \qquad a < \xi < b \tag{4}$$

so that

$$f'(\xi) = B \qquad a < \xi < b. \tag{5}$$

By solving equations (2) and (3) simultaneously for B, equation (5) becomes

$$f'(\xi) = \frac{f(b) - f(a)}{b - a} \qquad a < \xi < b \tag{6}$$

i.e.

$$f(b) = f(a) + (b - a)f'(\xi). \tag{7}$$

It is now natural to ask whether we can generalize the procedure adopted above by considering the difference between $f(x)$ and a quadratic approximation to $f(x)$. For this purpose we define

$$F(x) \equiv f(x) - (A_1 + B_1 x + C_1 x^2) \tag{8}$$

where A_1, B_1, and C_1 are constants which we wish to determine. Again we ask that the conditions $F(a) = F(b) = 0$ be imposed so that once more we may make use of Rolle's Theorem. There is certainly no loss in generality if we write (8) in the form

$$F(x) \equiv f(x) - [A + B(x - a) + C(x - a)^2] \tag{9}$$

which will be a little more convenient in simplifying the algebra involved. Upon imposing the condition that $F(a) = 0$ we find from (9) that $A = f(a)$, so that (9) becomes

$$F(x) \equiv f(x) - f(a) - B(x - a) - C(x - a)^2. \tag{10}$$

If $F(b)$ is to vanish we see from (10) that we must require the condition

$$f(b) = f(a) + B(b - a) + C(b - a)^2. \tag{11}$$

By applying Rolle's theorem we have

$$F'(\xi) \equiv f'(\xi) - B - 2C(\xi - a) = 0 \qquad a < \xi < b. \tag{12}$$

However this is not enough to determine the constants B and C. We do this by imposing a further condition. Since we have $F'(\xi) = 0$ it is natural to ask that $F'(a) = 0$, for if both $F'(a) = 0$ and $F'(a) = 0$ we will be able to apply Rolle's Theorem to the function $F'(x)$ and say that there exists a number m between a and ξ such that $F''(m) = 0$. Of course to do this we must be sure that $F'(x)$ is continuous in $a \le x \le b$ and has a derivative in $a < x < b$. This is accomplished by requiring that $f'(x)$ have similar properties. By asking that $F'(a) = 0$ we have from (10)

$$F'(a) \equiv f(a) - B = 0 \qquad \text{or} \qquad B = f'(a). \tag{13}$$

Thus (11) becomes

$$f(b) = f(a) + f'(a)(b - a) + C(b - a)^2. \tag{14}$$

Since $F'(a) = F'(\xi) = 0$ we apply Rolle's theorem to obtain

$$F''(m) = f''(m) - 2C = 0 \qquad a < m < \xi \tag{15}$$

or

$$C = \tfrac{1}{2}f''(m) \qquad a < m < \xi. \tag{16}$$

Thus (14) becomes

$$f(b) = f(a) + f'(a)(b - a) + \tfrac{1}{2}f''(m)(b - a) \qquad a < m < \xi < b. \tag{17}$$

This is merely a statement of Taylor's theorem with the remainder given in the form due to Lagrange. It is interesting to note that the number m in (17) is closer to a than the number ξ in (7).

Upon extending the above procedure to a consideration of the function

$$F(x) \equiv f(x) - [A + B(x - a) + C(x - a)^2 + D(x - a)^3] \tag{18}$$

where $f(x)$, $f'(x)$, $f''(x)$ are continuous in $a \le x \le b$ and $f'''(x)$ has a finite or infinite derivative in $a < x < b$, the reader may easily obtain, by successive applications of Rolle's theorem in a manner exactly analogous to that given above

$$f(b) = f(a) + f'(a)(b - a) + \frac{f''(a)}{2}(b - a)^2 + \frac{f'''(\lambda)}{6}(b-a)^3 \tag{19}$$

where $a < \lambda < m < \xi < b$.

Further extensions to the Taylor series with the remainder after any number of terms may easily be supplied by the reader.

One further remark may not be out of place here. Let us consider the function

$$F(x) \equiv f(x) - [A + B\,\theta(x)] \qquad a \le x \le b \tag{20}$$

where the given function $\theta(x)$ is used in place of x. By imposing the conditions $F(a) = F(b)$ we find

$$A + B\theta(a) = 0 \tag{21}$$

$$A + B\theta(b) = 0 \tag{22}$$

Hence by Rolle's Theorem (with the suitable restrictions on $f(x)$ and $\theta(x)$) we have

$$F'(\xi) = f'(\xi) - B\theta'(\xi) = 0. \tag{23}$$

Upon requiring that $\theta'(x) \neq 0$ in $a \le x \le b$ we have

$$\frac{f'(\xi)}{\theta'(\xi)} = B. \tag{24}$$

Hence from (21) and (22) if $\theta(a) \neq \theta(b)$ we have

$$\frac{f(b) - f(a)}{\theta(b) - \theta(a)} = \frac{f'(\xi)}{\theta'(\xi)} \qquad a < \xi < b. \tag{25}$$

Further generalizations may occur to the reader.

Rensselaer Polytechnic Institute
Hartford Graduate Center

On Finding the Characteristics of Logarithms

T. F. Mulcrone

The following method, probably not widely enough taught, of finding the characteristics of common logarithms, and of pointing off antilogarithms, replaces the usual set of involved rules by a single rule that is short and more easily remembered and applied.

The method supposes that numbers be written (or at least be regarded as if written) in "scientific notation", that is, as the product of a number between 1 and 10 (1 included) and an integral power of 10.

Thus

$$0.000000123 = 1.23 \ (10^{-7})$$
$$1.23 = 1.23 \ (10^{0})$$
$$1230 = 1.230 \ (10^{3})$$

Then we have, for any number in scientific notation, v.g. $1.23 \ (10^n)$,

$$\log 1.23 \ (10^n) = \log 1.23 + \log 10^n = \log 10^n + \log 1.23 = n + 0.08991$$

and we note the following

RULE: The characteristic of $\log_{10} N$ *is equal to the exponent of 10 in the scientific notation representation of N.*

The rule is applied as in the following examples:

(i) $\log 1230 = \log 1.230 \ (10^3) = 3.08991;$

(ii) $\log 0.000000123 = \log 1.23 \ (10^{-7}) = \overline{7}.08991 = 3.08991 - 10;$

(iii) If $\log N = 7.08991$, then $N = 1.23 \ (10^7) = 12300000;$

(iv) If $\log N = -3.91009$, then $\log N = 10.00000 - 10 - 3.910009 = 6.08991 - 10 = 4.08991$

Hence $N = 1.23 \ (10^{-4}) = 0.000123$.

St. Charles College, Grand Coteau, La.

MISCELLANEOUS NOTES

Edited by

Charles K. Robbins

Articles intended for this Department should be sent to Charles K. Robbins, Department of Mathematics, Purdue University, Lafayette, Indiana.

INNER PRODUCT OF TWO CIRCLES

A. R. Amir-Moéz

Circles of a given plane, in some respects, behave like vectors. The idea is due to Professor M. Hachtrouch, University of Teheran. The inner product of two circles is interesting to study in addition to its being a powerful tool for writing the equations of lines and circles mutually tangent or intersecting in a given angle α.

I. **Definition**: Let C_1 and C_2 be two circles with the following equations:

(1) $a_1(x^2 + y^2) - b_1x - c_1y + d_1 = 0, \quad a_1 > 0$

(2) $a_2(x^2 + y^2) - b_2x - c_2y + d_2 = 0, \quad a_2 > 0$

We define (C_1, C_2), the inner product of (1) and (2), to be $a_1R_1 \cdot a_2R_2 \cos\alpha$ where R_1 and R_2 are respectively the radii of C_1 and C_2 and α is the angle between C_1 and C_2. a_1R_1 and a_2R_2 are defined to be the absolute values of C_1 and C_2 respectively.

II. **Theorem**: Let C_1 and C_2 be given by (1) and (2). Then

$$(C_1, C_2) = b_1b_2 + c_1c_2 - \frac{a_2d_1 + a_1d_2}{2}, \text{ and}$$

the absolute value of C_1 is $a_1R_1 = \sqrt{b_1^2 + c_1^2 - a_1d_1}$.

Proof: Let $f(x,y) = 0$ and $g(x,y) = 0$ be the equations of two plane curves such that $\frac{\partial f}{\partial x}, \frac{\partial g}{\partial y}, \frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}$ exist at (x_0, y_0), one of the points of the intersection of f and g. Then for α, the angle between f and g at (x_0, y_0), we have

$$(3) \quad \cos\alpha = \frac{\frac{\partial f}{\partial x}\frac{\partial g}{\partial x} + \frac{\partial f}{\partial y}\frac{\partial g}{\partial y}}{\sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2} \cdot \sqrt{\left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2}}$$

This is a well known fact of differential geometry. Applying (3) to (1) and (2), the proof of the theorem is easily seen.

The theorem holds in the case of complex points of intersection. In this case $|\cos\alpha|$ will be greater than one and is equal to $\left|\frac{e^{i\alpha}+e^{-i\alpha}}{2}\right|$ where α is a complex number. This is a useful fact for determining whether two circles intersect or not. We shall give an example at the end of this discussion.

III. **Remark:** $px + qy + d = 0$ can be considered as a circle where

$$a = 0, \quad b = \frac{-p}{2}, \quad c = \frac{-q}{2}, \text{ and } d = d. \text{ Clearly from II}$$

$$ar = \frac{\sqrt{p^2+q^2}}{2}$$

IV. **Example:** Let

(4) $x^2 + y^2 - x - y - 3 = 0$ be given Find a relation among p, q, and r so that $px + qy + r = 0$ be tangent to (4).

Solution: For (4) we have

$a_1 = 1,\ b_1 = 3,\ c_1 = 2,\ d_1 = -3;$ and for the line

$a_2 = 0,\ b_2 = -p/2,\ c_2 = -q/2,\ d_2 = r.$

Since the line is to be tangent to (4) $\cos\alpha = 1$. Therefore by I, II, and III we have

$$a_1R_1 \cdot a_2R_2 \cos\alpha = \frac{-3p}{2} - q - \frac{r}{2} = 4\,\frac{\sqrt{p^2+q^2}}{2}$$

V. **Example:** Let $x^2 + y^2 - 2x = 0$ and $x^2 + y^2 + 4x + 3 = 0$ be two circles. Clearly

$a_1 = 1,\ b_1 = 1,\ c_1 = 0,\ d_1 = 0$

$a_2 = 1,\ b_2 = -2,\ c_2 = 0.\ d_2 = 3$

By II $\cos\alpha = \frac{-7}{2}$. This means the two circles do not intersect.

University of Idaho.

A Method for Solving $\int \sin^{2n}\alpha x\, dx$ and $\int \cos^{2n}\alpha x\, dx$

L. L. Pennisi

The usual methods for evaluating

(1) $$\int \sin^{2n}\alpha x\, dx \quad \text{and} \quad \int \cos^{2n}\alpha x\, dx$$

are based upon the repeated applications of the trigonometric identities

$$\sin^2 A = \tfrac{1}{2}(1 - \cos 2A) \quad \text{and} \quad \cos^2 A = \tfrac{1}{2}(1 + \cos 2A).$$

An alternative approach is to utilize the following identity,

(2) $$e^{ix} = \cos x + i \sin x,$$

where x is a real number and $i = \sqrt{-1}$, (see, C. Palmer and C. Leigh, Plane and Spherical Trigonometry, pp. 184-185).

Then for any real number α, we have from (2)

(3a) $$\sin \alpha x = \frac{e^{i\alpha x} - e^{-i\alpha x}}{2i}$$ and (3b) $$\cos \alpha x = \frac{e^{i\alpha x} + e^{-i\alpha x}}{2}.$$

Hence, for any positive integer n, when we apply the binomial expansion to the right member of (3a) and then use (3b) we obtain,

(4) $$\sin^{2n}\alpha x = (4)^{-n}\binom{2n}{n} + 2(-4)^{-n}\sum_{k=0}^{n-1}(-1)^k\binom{2n}{k}\cos 2(n-k)\alpha x,$$

where

(5) $$\binom{2n}{k} = \begin{cases} \dfrac{2n(2n-1)\ldots(2n-k+1)}{k!}, & k = 1, 2, \ldots, n, \\ 1, \text{ for } k = 0. \end{cases}$$

Integrating (4) with respect to x, we obtain

(6) $$\int \sin^{2n}\alpha x\, dx = (4)^{-n}\binom{2n}{n}x + (-4)^{-n}\sum_{k=0}^{n-1}\frac{(-1)^k\binom{2n}{k}\sin 2(n-k)\alpha x}{(n-k)\alpha} + C,$$

where C is a constant of integration.

If we use the binomial expansion on the right member of (3b), and then proceed as above, we find,

(7) $$\int \cos^{2n}\alpha x\, dx = (4)^{-n}\left[\binom{2n}{n}x + \sum_{k=0}^{n-1}\frac{\binom{2n}{k}\sin 2(n-k)\alpha x}{(n-k)\alpha}\right] + C.$$

REMARK. Having established (6) one may evaluate,

$$\int \sin^{2m} ax \quad \cos^{2n} ax \; dx.$$

However, the evaluation of this integral leads to an expression which cannot be readily computed, as was the case for the integrals (6) and (7). To evaluate it, we proceed as follows:

(8)
$$\sin^{2m} ax \cos^{2n} ax = \sin^{2m} ax(1 - \sin^2 ax)^n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \sin^{2(k+m)} ax.$$

Using (8) and (6), we obtain,

(9)
$$\int \sin^{2m} ax \cos^{2n} ax \, dx = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \left[4^{-(k+m)} \binom{2k + 2m}{k + m} x \right.$$

$$\left. + (-4)^{-(k+m)} \sum_{j=0}^{k+m-1} \frac{(-1)^j \binom{2k + 2m}{j} \sin 2(k + m - j) ax}{(k + m - j) a} \right] + C.$$

University of Illinois

An Application of a Cauchy Functional Equation in Elementary Differential Equations

Roy F. Reeves and Lyle E. Pursell

Many physical applications of elementary differential equations lead to the differential equation $y' = my$ where m may or may not be known. Unfortunately the transition from the physical problem to the differential equation is frequently effected by arguments involving "rate of change" or approximation by differentials which the student finds elusive. In this note we give a way of handling these problems which is more involved mathematically, but which places less stress on intuition.

As an example consider the problem of finding the amount of light transmitted by a pane of glass as a function of the thickness T (cf. Agnew, *Differential Equations*, pp. 41-43). We choose our coordinate system in such a way that the surface of incidence is the plane $x = 0$ and the surface of emission is the plane $x = T$. Consider any two planes $x = x_0$ and $x = x_1$ in the glass such that $0 \leq x_0 < x_1 \leq T$. Let the probability of a light particle reaching the plane x be $p(x)$ under the assumption that the glass is homogeneous and that failure of a

particle to reach the plane x is due only to its striking a glass molecule. Under these assumptions it follows that the probability of a particle reaching the plane $x = x_1$ granted that it has reached the plane $x = x_0$ is $p(x_1 - x_0)$.
Hence

(1) $$p(x_1) = p(x_0)\, p(x_1 - x_0).$$

Let $x_1 - x_0 = y$ and $x_0 = x$. Then

(2) $$p(x + y) = p(x)\, p(y).$$

If we assume the light to be of macroscopic intensity throughout the pane, then the amount $A(x)$ of light reaching the plane x may be taken to be $A(x) = A(0)\, p(x)$ and a reasonably realistic solution may be expected to follow the assumption that $A(x)$ is continuous. It is well known that all continuous solutions of (2) are of the form $p(x) = \exp\,(ax)$ where a is a constant (Courant, *Differential and Integral Calculus,* vol. I, pp. 169-173). Hence $A(x) = A(0) \exp\,(ax)$ where a would be determined experimentally. In a differential equations class if one assumes $p(x)$ is differentiable at $x = 0$, then (since $p(0) = 1$)

$$\frac{p(x+h) - p(x)}{h} = \frac{p(x)\,[p(h) - 1]}{h} = \frac{p(x)\,[p(h) - p(0)]}{h}.$$

Taking limits as h goes to zero, we have,

$$p'(x) = p(x)p'(0).$$

Hence $p(x) = \exp\,[p'(0)x]$ and $A(x) = A(0) \exp\,[p'(0)x]$.

In a number of other problems (e.g. problems of bacterial growth and of continuously compounded interest) involving a quantity y dependent on a variable x, the ratio $y(x_1)/y(x_0)$ is seen to depend only on $(x_1 - x_0)$, but it is not so obvious that $y'(x)$ is proportional to $y(x)$. If one sets $f(x_1 - x_0) = y(x_1)/y(x_0)$ then

$$f(x_0)\, f(x_1 - x_0) = \frac{y(x_0)}{y(0)} \cdot \frac{y(x_1)}{y(x_0)} = \frac{y(x_1)}{y(0)} = f(x_1).$$

Hence $f(x)$ satisfies equation (1) and (2) and one may show as above $f(x) = \exp\,(ax)$. Hence $y(x) = y(0) \exp\,(ax)$.

The Ohio State University.

A Commission on Mathematics

A Commission on Mathematics that will investigate the need for revision of the secondary school mathematics curriculum has been appointed by the College Entrance Examination Board.

New frontiers in mathematics are creating almost unlimited opportunities for growth in mathematical knowledge and its applications to physical science and engineering, to the social and biological sciences, and to business and industry. The Commission has been established by the College Board in recognition of a growing divergencesbetween these developments and the type of mathematics taught in secondary school.

The Commission on Mathematics is a group of 13 high school and college mathematics teachers. Its charge is broad in scope. Several regional conferences of high school and college teachers and administrators will be summoned to advise and assist the Commission. Following such meetings, and in the light of the experience and deliberations of its own members, the Commission will recommend necessary and desirable action to the College Board.

At a meeting held at Princeton, New Jersey, in January, 1956, the Commission took some initial steps in its task of pondering desirable changes. One of the results of this meeting was the organization of several subcommittees to carry on the Commission's activities.

The Commission has established liaison with other groups undertaking work related to its own. These include committees of the National Council of Teachers of Mathematics, the Mathematical Association of America, the American Society for Engineering Education, and the American Association for the Advancement of Science.

The Commission has scheduled meetings for May and October, 1956, and tentative plans have been made for summer writing groups to develop sample materials within teaching units that might be recommended for inclusion in the high school curriculum.

Members of the Commission are: A. W. Tucker of Princeton, chairman; C. B. Allendoerfer of the University of Washinton; E. C. Douglas of Watertown, Connecticut; H. F. Fehr of Teachers College, Columbia; M. Hildebrandt of Maywood, Illinois; A. E. Meder of Rutgers: F. Mosteller of Harvard; E. P. Northrop of the University of Chicago; E. R. Ranucci of Newark, New Jersey; R. E. K. Rourke of Kent, Connecticut; G. B. Thomas of MIT; H. Van Engen of Iowa State Teachers College; and S. S. Wilks of ETS and Princeton.

Anyone interested in being kept informed of Commission progress should write Mr. Robert Kalin, Executive Assistant, Commission on Mathematics, P.O. Box 592, Princeton, New Jersey.

INTEGRAL TWINS.

Bancroft H. Brown

Which of the following triangles has the larger inscribed circle: one with sides 17, 25, and 26; or one with sides 17, 25, and 28?

The formula

(1) $r^2 s = (s-a)(s-b)(s-c)$, where $2s = a+b+c$

yields the somewhat unexpected fact that in each case the radius is 6.

DEFINITIONS: Four positive numbers satisfying (1) are called a *quadruple* (a, b, c, r). With a quadruple is associated a real triangle, and conversely. If all four numbers are integers, it is an *integral quadruple*. Quadruples (a, b, c, r) and (a, b, c', r) are called *twins*. Twins which are both integral quadruples are called *integral twins*. Integral twins for which the five numbers have no common factor are called *primitive*. Multiples of primitives are a rather trivial generalization.

PROBLEM: to find integral twins. The above example shows their existence. Apparently nothing has appeared in the literature on the subject. If, in (1), we are given a, b, and r, then (1) is a cubic in c. It is easy to show that one root is always negative, so that *triplets* do not exist. But, in general, every quadruple has a twin, although in rare and not particularly interesting cases $c' = c$ and the twins are *identical*. It is, of course, unusual for (a, b, c, r) to be an *integral* quadruple. But if it is, it follows that (a, b, c', r) is also an integral quadruple if and only if

(2) $$\frac{\{16r^2 + r(a-b)^2\}\,(a+b)}{c} + (a+b-c)^2$$

is a perfect square, say T^2, in which case $c' = (T + a + b - c)/2$.

While there is an immense amount of literature on integral quadruples, none of it seems to help in any way to solve our problem. Thus for right and for isosceles triangles, we know all possible integral quadruples, but it can be shown that no one of these possesses an integral twin. (This is true whether "c" is chosen as a leg or as the hypotenuse of a right triangle, and whether it is one of the equal sides or the unequal side of an isosceles triangle.) This negative fact is also true for other classic families of integral quadruples; apparently twins do not occur in well-ordered families, but involve a certain randomness.

One method of attack is to note in (1) that s must be an integer and then to replace $(s-a)$ by A, etc., giving

$$r^2(A + B + C) = ABC. \tag{3}$$

For an assigned r, not too large, it is tedious, but not impossible to enumerate all the positive integral solutions of (3). The standard integral quadruples (a, b, c, r) are then easily obtained and, if integral twins occur, they will be immediately spotted. The triangles associated with twins may be both acute $(A - A)$, one acute and one obtuse $(A - 0)$, or both obtuse $(0 - 0)$; however, the case $(0 - 0)$ is apparently very rare. Now it is very much easier to enumerate the *acute* solutions of (3) than the *obtuse*, since the acute solutions have the property that A, B, and C are all larger than r. Thus, if we pass up the not very promising $(0 - 0)$ field, the labor is cut down enormously by taking the very few acute integral quadruples which occur and testing them by (2) for integral twins. The profitable fields are those in which r is composite, particularly if r is a multiple of 6. Certain barren deserts are indicated by the following very curious

Theorem: Integral, acute, scalene triangles with *prime* r do not exist.

The following table indicates all the *primitive* integral twins that have been found after a fairly protracted search. The fourth entry is a bit of a tour de force, and is the result of a sustained effort to find out whether $(0 - 0)$ twins could exist. The author would welcome additions. He thinks it possible that the list is complete for $r < 72$.

a	b	c	c'	r	Nature
17	25	26	28	6	$A - A$
41	50	39	73	12	$A - 0$
65	68	57	105	18	$A - 0$
97	169	122	228	30	$0 - 0$
97	340	339	345	42	$A - A$
100	291	289	299	42	$A - A$
255	260	103	485	42	$A - 0$

Dartmouth College

A GEOMETRY PROBLEM

H. F. Mathis

The circle C_1 encloses the circle C_2 as shown in Fig. 1. The centers of these circles are 0_1 and 0_2. The points a and b are arbitrary points on C_1 and C_2, respectively, which satisfy the conditions that both points lie on the same side of the line $0_1 0_2$ and the lines ab and $0_1 0_2$ are perpendicular. The problem is to determine the location of the point c, which is the intersection of the lines ab and $0_1 0_2$, so that $\overline{ab}/\overline{bc}$ has its minimum value. This is equivalent to locating c so that the tangents to the circles through the points a and b intersect on the line $0_1 0_2$.

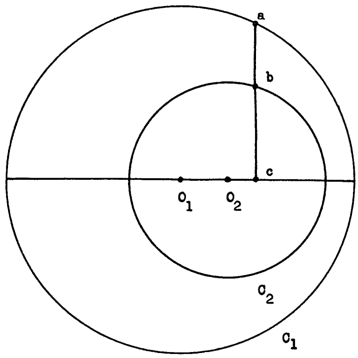


Fig. 1.

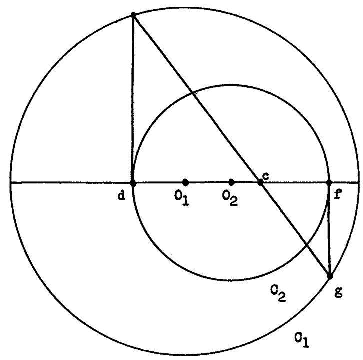


Fig. 2.

A simple procedure for locating c is illustrated in Fig. 2. The lines de and fg are tangent to C_2 and perpendicular to the line $0_1 0_2$. The intersection of the lines eg and $0_1 0_2$ is the desired location of the point c.

In order to verify that this solution is correct, it is convenient to use a rectangular coordinate system with the origin at 0 and the x-axis along the line $0_1 0_2$. Let $x_0 = \overline{0_2 c}$, $p = \overline{0_1 0_2}$, r_1 = the radius of C_1, and r_2 = the radius of C_2. It is easily seen that $\overline{ab}/\overline{bc}$ is a minimum when $\overline{ac}^2/\overline{bc}^2$ is a minimum. Since

$$\overline{ac}^2 = r_1^2 - (x_0 + p)^2$$

and

$$\overline{bc}^2 = r_2^2 - x_0^2,$$

the problem is to determine the value of $x = x_0$ which minimizes

$$F(x) = \frac{r_1^2 - (x + p)^2}{r_2^2 - x^2}.$$

It can be shown that if $0 < x_0 < r_2$ and x_0 is a root of the equation

(1) $$px^2 - (r_1^2 - r_2^2 - p^2)x + r_2^2 p = 0,$$

then

$$\frac{dF(x_0)}{dx} = -\frac{2[px_0^2 - (r_1^2 - r_2^2 - p)x_0 + r_2^2 p]}{(r_2^2 - x_0^2)^2} = 0$$

and

$$\frac{d^2F(x_0)}{dx^2} = \frac{2p}{x_0(r_2^2 - x_0^2)} > 0.$$

Both roots of equation (1) are positive. Only one root need be considered because if $x_0 < r_2$, then the other root is greater than r_2 since the product of the two roots is r_2^2. It is obvious that the second root cannot be a solution. Consequently, x_0 is the desired solution.

Let x_1 denote the intersection of lines eg and $0_1 0_2$. Since the triangles x_1de and x_1fg are similar,

(2) $$\frac{\overline{de}^2}{\overline{dx_1}^2} = \frac{\overline{fg}^2}{\overline{x_1f}^2} = \frac{r_1^2 - (p - r_2)^2}{(r_2 + x_1)^2} = \frac{r_1^2 - (p + r_2)^2}{(r_2 - x_1)^2}.$$

Equation (2) can be converted to equation (1). Therefore, $x_1 = x_0$, and the location of c given by the graphical procedure is the same as that which minimizes $\overline{ab}/\overline{bc}$.

The equation of the tangent to circle C_1 through a is

$$\frac{x - x_0}{\overline{ac}} + \frac{y - \overline{ac}}{p + x_0} = 0.$$

This line intersects the x-axis at the point

$$x_2 = x_0 + \frac{ac^2}{p + x_0}.$$

The equation of the tangent to C_2 through b is

$$\frac{x - x_0}{\overline{bc}} + \frac{y - \overline{bc}}{x_0} = 0.$$

This line intersects the x-axis at the point

$$x_3 = x_0 + \frac{\overline{bc}^2}{x_0}.$$

Since

$$\overline{ac}^2 = r_1^2 - (p + x_0)^2,$$

$$\overline{bc}^2 = r_2^2 - x_0^2,$$

and x_0 satisfies equation (1), it follows that

$$\frac{\overline{ac}^2}{p + x_0} = \frac{\overline{bc}^2}{x_0}$$

and

$$x_2 = x_3 .$$

Therefore, the two tangents intersect on the line $0_1 0_2$.

Goodyear Aircraft Corp., Akron, Ohio.

PROBLEMS AND QUESTIONS

Edited by

Robert E. Horton, Los Angeles City College

Readers of this department are invited to submit for solution problems believed to be new and subject matter questions that may arise in study, in research, or in extra-academic situations. Proposals should be accompanied by solutions, when available, and by any information that will assist the editor. Ordinarily, problems in well-known textbooks should not be submitted.

Solutions should be submitted on separate, signed sheets. Figures should be drawn in India ink and twice the size desired for reproduction.

Send all communications for this department to *Robert E. Horton, Los Angeles City College, 855 North Vermont Ave., Los Angeles 29, California.*

PROPOSALS

271. *Proposed by C. W. Trigg, Los Angeles City College.*

In the scale of 8, the addition $ATOM + BOMB = CHAOS$ utilizes a $BOMB \equiv 1 \pmod 7$ and produces a $CHAOS$ which is a permutation of consecutive digits. Each letter uniquely represents a digit. Convert the letters into digits.

272. *Proposed by M. Rumney, London, England.*

Given three different positive integers N_1, N_2, N_3. Find a partition of N_1 into three different positive integers a_{11}, a_{12}, a_{13}; N_2 into three different positive integers a_{21}, a_{22}, a_{23} and N_3 into three different positive integers a_{31}, a_{32}, a_{33} such that

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = 0$$

273. *Proposed by N. A. Court, University of Oklahoma.*

The points of intersection of the tangents to the circumcircle of a triangle drawn at the ends of one side is collinear with the two points which that circle marks on the median issued from the opposite vertex and on the parallel through that vertex to the side considered.

274. *Proposed by Monte Dernham, San Francisco, California.*

Tom and Jerry together have m dollars. Tom gives Jerry as much money as Jerry already has. Then Jerry gives back to Tom as much as Tom has left, whereupon Tom gives Jerry as much as Jerry then has, and so on. After exactly n such transfers of cash, each of which serves to double the recipient's then existing pile, one of the two participants finds he has nothing left. (1) Which one? (2) How much did each have at the start? (3) What restriction on m is necessary and sufficient to justify the assumption that each started with an integral number of dollars?

275. *Proposed by L. Carlitz, Duke University.*

I. Let the prime $p = 3m + 1$. Show that

$$\sum_{r=0}^{m-1} \binom{3r+1}{r} \left(\frac{1}{9}\right)^r \equiv \begin{cases} -3 & (m \equiv 1 \pmod 3) \\ 3 & (m \equiv -1 \pmod 3) \\ 0 & (m \equiv 0 \pmod 3) \end{cases} \pmod p$$

$$\sum_{r=0}^{m-1} \binom{3r+1}{r} \left(\frac{2}{27}\right)^r = 0 \pmod p.$$

II. Let the prime $p = 4m + 1$. Show that

$$\sum_{r=0}^{m-1} \binom{4r+1}{2r} \left(\frac{1}{16}\right)^r \equiv (-1)^{m-1} \pmod p.$$

276. *Proposed by P. H. Yearout, Portland, Oregon.*

Solve simultaneously

$$x + y + z = 2$$
$$x^2 + y^2 + z^2 = 14$$
$$x^4 + y^4 + z^4 = 98$$

277. *Proposed by J. M. Howell, Los Angeles City College.*

In a game, A has probability 2/3 of winning \$1, probability 5/24 of losing \$1, and probability 1/8 of losing \$2. A starts with \$10 and his opponent has unlimited resources. What is the probability of A's wealth not decreasing to \$1 or less? We assume that A must stop if he has only \$1, since he could not pay a \$2 loss.

SOLUTIONS

Late Solutions

245. *K.L.Cappel, The Franklin Institute, Philadelphia, Pennsylvania.*

248. *Major H.S.Subba Rao, Defense Science Organization, New Delhi, India.*

Number Partitions

228. [January 1955] *Proposed by Howard D. Grossman, New York, New York.*

Prove that the number of partitions of any number into odd parts greater than unity is equal to the number of partitions into two or more unequal parts of which the two largest differ by unity.

Solution by Richard K. Guy, University of Malaya, Singapore.

Consider
$$\frac{1}{(1-x^3)(1-x^5)(1-x^7)\ \dots} = \sum \mathfrak{f}_1(m)x^n$$

as the generating function for $\mathfrak{f}_1(m)$, the number of partitions of n into odd parts greater than 1.

$(1+x)(1+x^2)(1+x^3)\ \dots = \sum \mathfrak{f}_u(n)x^n$ is the generating function for $\mathfrak{f}_u(n)$, the number of partitions of n into unequal parts.

Lemma $\mathfrak{f}_u(n) - \mathfrak{f}_u(n-1) = \mathfrak{f}_2(n)$, where $\mathfrak{f}_2(n)$ is the number of partitions of n into unequal parts of which the two greatest differ by 1.

Proof: In the graphical representation of a partitions into unequal parts, the first two rows differ by more than one as in

a)
```
x x x x x x x x x x (x)
x x x x x x x
x x x x
x
```

or by exactly one as in

b)
```
x x x x x x x x
x x x x x x x
x x x x
x x
x
```

In the former case we can remove the top right hand member and produce a partition of $n-1$ into unequal parts. In the latter case we cannot. This establishes a one to one correspondence between the partitions $\mathfrak{f}_u(n)$ and either the partitions of $\mathfrak{f}_u(n-1)$ or the partitions $\mathfrak{f}_2(n)$. That is $\mathfrak{f}_u(n) = \mathfrak{f}_u(n-1) + \mathfrak{f}_2(n)$.

Next we prove the main theorem $\mathfrak{f}_2(n) = \mathfrak{f}_1(n)$. As

$$(1-x)(1+x)(1+x^2)(1+x^3)\ \dots =$$

$$(1-x)\,\frac{(1-x^2)}{(1-x)}\cdot\frac{(1-x^4)}{(1-x^2)}\cdot\frac{(1-x^6)}{(1-x^3)} = \frac{1}{(1-x^3)(1-x^5)(1-x^7)\ \dots}.$$

Here the right hand side is the generating function $f_1(n)$. The coefficient of x^n on the left hand side is $f_u(n) - f_u(n-1)$ which by the lemma is $f_2(n)$ so $f_2(n) = f_1(n)$.

The Game of Bowling

250. [November 1955] *Proposed by V. C. Harris, San Diego State College California.*

In how many different ways can the game of bowling take place? Consider only the number of pins knocked down by each ball, not the particular position of the pins.

Solution by M. J. Pascual, Siena College, New York. For the first nine frames there are three mutually exclusive and exhaustive possibilities, namely a strike, a spare or a miss. These may be obtained in one, ten and fifty five different ways respectively. Hence, the first nine frames can be bowled in 66^9 different ways. The tenth frame can be bowled in 241 different ways; 55 ways where neither spare or strike is made, 110 ways where a spare is made and 76 ways where a strike is made. Thus, the total number of ways is $(241)66^9$ or 5,726,805,883,325,784,576.

Also solved by the proposer. Two incorrect solutions were received.

Related Quadratics

251. [November 1955] *Proposed by George U. Brauer, University of Minnesota.*

Find all integers a and b such that $x^2 + ax + b = 0$ and $x^2 + bx + a = 0$ have integral roots.

Solution by E. P. Starke, Rutgers University. The equations (1) $x^2 + ax + b = 0$ and (2) $x^2 + bx + a = 0$ have all integral roots for the values (a,b), $(4,4)$, $(5,6)$, $(-t, t-1)$, $(0, -t^2)$ and for the pairs in which a and b are interchanged; t is any integer. To prove there are no other values we use the following:

Lemma: The equation $x^2 + px + q = 0$, $q \neq 0$, has a root r with $|r| < 1$ (and hence not integral) unless

(3) $q \geq |p| - 1$ for $q > 0$, or $-q \geq |p| + 1$ for $q < 0$.

These follow easily from the inequality $|-p \pm \sqrt{p^2 - 4q}| < 2$.

Of course, a and b must be integers if all the roots are integers In case $a > 0$ and $b > 0$ (3) requires $b \geq a - 1$ and $a \geq b - 1$, whence $b - a = 0, \pm 1$. Now if $b - a = 0$, then in order that the discriminant of (1) be a square, $a^2 - 4a = r^2$ say, or $(a-2)^2 - r^2 = 4$, we must have $r = 0$ and $a - 2 = \pm 2$. Similarly, if $b - a = 1$, we have $(a - 2)^2 - r^2 = 8$, so that $r = 1$, $a - 2 = \pm 3$; $b - a = -1$ gives the same with a, b interchanged.

The case $a < 0$ and $b < 0$ cannot occur since (3) requires $-b \geq -a + 1$ and $-a \geq -b + 1$, which are contradictory.

In case $a < 0$ and $b > 0$, (3) gives $b \geq -a - 1$, $-a \geq b + 1$, whence we must have $a + b = -1$. By symmetry, the same condition results from $a > 0$ and $b < 0$.

These, together with the obvious condition corresponding to $a = 0$ (or $b = 0$), give the results stated at the outset.

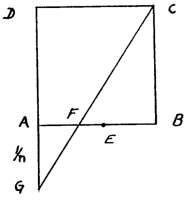
Also solved by Richard K. Guy, University of Malaya, Singapore; Sam Kravitz, East Cleveland, Ohio; F. W. Saunders, Coker College, South Carolina and Chih-yi Wang, University of Minnesota.

A Construction of 1/n

252. [November 1955] *Proposed by Donald M. Brown, Willow Run Research Center.*

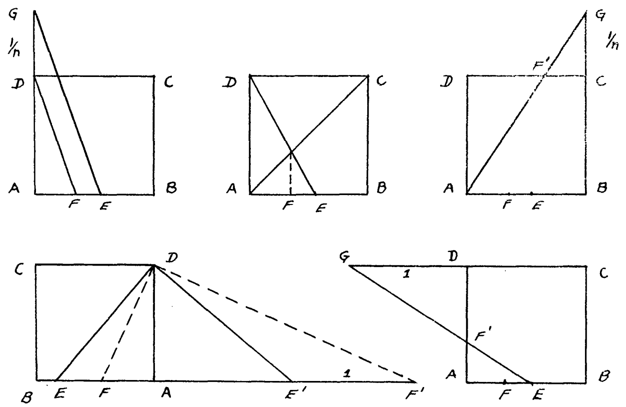
Given a square with vertices $ABCD$ with sides of unit length, and a point E on side AB such that $AE = 1/n$ units. Devise a simple geometric construction to locate the point F such that $AF = 1/(n + 1)$ units.

Solution by Mary Ann Long, Student at Winthrop College, South Carolina. Extend DA to a point G so that AG equals AE. Draw GC and let F denote its intersection with AB. Then from similar Triangles $AF = 1/(n + 1)$.



Editor's note: Using Lemoine's Criteria the solution above was the simplest received, (S9, E6). It was submitted by several solvers.

Also solved by Leon Bankoff, Los Angeles, California; J. W. Clawson, Collegeville, Pennsylvania; Raphael R. Coffman, Richland, Washington; Huseyin Demir, Zonguldak, Turkey; H.M.Feldman, Washington University, St. Louis, Missouri; R. K. Guy, University of Malaya, Singapore; M. A. Kirchberg, Milwaukee, Wisconsin; Joseph D. E. Konhauser; State College, Pennsylvania; Thomas F. Mulcrone, St. Charles College, Louisiana; Michael Pascual, Siena College, New York; F. W. Saunders, Coker College, South Carolina; P. D. Thomas, Eglin Air Force Base, Florida; Alan Wayne, Cooper Union School of Engineering, New York; Chih-yi Wang, University of Minnesota and the proposer. The five essentially different kinds of constructions are shown below.



Not so Persistent Number

253. [November 1955] *Proposed by M. S. Klamkin and C. H. Pearsall Jr. Polytechnic Institute of Brooklyn.*

In Ripley's (New) "Believe It Or Not" the following statement appears (p. 207). "The Persistent Number 526, 315, 789, 473, 684, 210 may be multiplied by any number. The original digits will always reappear in the result." Show that this statement is not correct.

Solution by M. A. Kirchberg, Milwaukee, Wisconsin. Observing that twice this number equals one-tenth of it less 1 plus 10^{18}, we see that 19 times the number is $10^{19}-10$ or 9999999999999999990.

Also solved by R. K. Guy, University of Malaya, Singapore; F. W. Saunders, Coker College, South Carolina; Alan Wayne, Cooper Union School of Engineering, New York, Chih-yi Wang, University of Minnesota and the proposers.

A Finite Limit

254. [November 1955] *Proposed by Barney Bissinger, Lebanon Valley College.*

If $S_1 = k \geq 1$ and $S_{n+1} = 4S_n^3 - 3S_n$ for $n \geq 1$ find $\lim_{n\to\infty} \dfrac{S_{n+1}}{\prod_1^n (4S_i^2-1)}$.

I. *Solution by Major H. S. Subba Rao, Defense Science Laboratory, New Delhi, India.* We begin with

$$S_{i+1} - 1 = 4S_i^3 - 3S_i - 1$$
$$= (S_i - 1)(2S_i + 1)^2$$

and
$$S_{i+1} + 1 = (S_i + 1)(2S_i - 1)^2$$

Hence

$$\frac{S_{i+1}^2 - 1}{S_i^2 - 1} = (4S_i^2 - 1)^2, \qquad i = 1, 2, 3, \ldots n.$$

Therefore

$$\prod_{i=1}^{n} \left[\frac{S_{i+1}{}^2 - 1}{S_i^2 - 1}\right] = \prod_{i=1}^{n} (4S_i^2 - 1)^2 .$$

That is

$$\frac{S_{n+1}{}^2 - 1}{S_i^2 - 1} = \prod_{i=1}^{n} (4S_i^2 - 1)^2$$

or

$$\frac{S_{n+1}}{\prod_{i=1}^{n} (4S_i^2 - 1)^2} - \frac{1}{\prod_{i=1}^{n} (4S_i^2 - 1)^2} = S_i^2 - 1 = K^2 - 1.$$

Now because $\prod_{i=1}^{n} (4S_i^2 - 1)^2$ is a polynomial in K whose degree increases with n and hence $\lim_{n\to\infty} \frac{1}{\prod_{i=1}^{n} (4S_i^2 - 1)^2} = 0$.

Since $K \geq 1$ we have the conclusion $\lim_{n\to\infty} \frac{S_{n+1}}{\prod_{i=1}^{n} (4S_i^2 - 1)} = \sqrt{K^2 \mp 1}$.

II. *Solution by H. M. Feldman, Washington University, St. Louis, Missouri.* Let $K = (e^{\theta} + e^{-\theta})/2 = \cosh \theta$, then it is easily seen that

$$S_{i+1} = \cosh 3^i\theta \text{ and } 4S_i^2 - 1 = 4 \cosh^2(3^{i-1}\theta) - 1 = \frac{\sinh 3^i \theta}{\sinh 3^{i-1} \theta}.$$

We thus get $\prod_1^n (4S_i^2 - 1) = \prod_1^n \frac{\sinh 3^i \theta}{\sinh 3^{i-1}\theta} = \frac{\sinh 3^n \theta}{\sinh \theta}$.

From this we obtain

$$\lim_{n\to\infty} \frac{S_{n+1}}{\prod (4S_i^2 - 1)} = \lim_{n\to\infty} \frac{\cosh 3^n \theta}{\sinh 3^n \theta / \sinh \theta} = \sinh \theta = \sqrt{K^2 - 1}$$

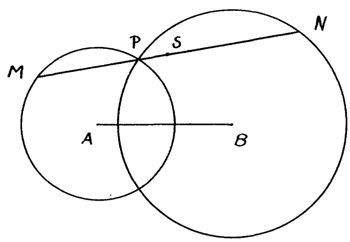
Also solved by Richard K. Guy, University of Malaya, Singapore; Chih-yi Wang, University of Minnesota and the proposer.

Equal Circumcircles

255. [November 1955] *Proposed by Leon Bankoff, Los Angeles, California.*

Two orthogonal circles, centers at A and B, intersect in P and Q. A line through P cuts circle A in M and circle B in N. S is the mid-point of MN. Show by elementary geometry that the nine-point circles of triangles ABP and ABS are equal.

Solution by Huseyin Demir, Zonguldak, Turkey. Let the midpoints of PM, PN, PS, be M', N', S' respectively. Since S is the midpoint of MN it follows that S' is also midpoint of $M'N'$. Hence the perpendicular erected at S' to MN bisects AB in O. This shows that S and P are equidistant from O. On the other hand we have $OA = OB = OP$ from the orthogonality of (A) and (B). This proves that ASB and APB have the same circumcircle. This justifies the statement.



Also solved by George T. Apostolopoulos, Wright Junior College, Chicago, Illinois; J. W. Clawson, Collegeville, Pennsylvania; Richard K. Guy, University of Malaya, Singapore; F. W. Saunders, Coker College, Hartsville, South Carolina; P.D.Thomas, Elgin Air Force Base, Florida; Chih-yi Wang, University of Minnesota and the proposer.

A Vector Space Function

256. [November 1955] *Proposed by Maïmouna Edy, Hull, P. Q. Canada.*

Let $u = u(t)$ be a p- times differentiable function from an interval (a, b) to an N-dimensional real vector space. The successive derivatives of u are denoted by u', u'', ..., $u^{(p)}$. Let the first $(p-1)$ of these derivatives be linearly independent while the first p of them are linearly dependent for all t in (a, b). Show that u belongs to a fixed subspace of dimension p.

Solution by Chih-yi Wang, University of Minnesota. In this solution if we say a polynomial of degree n, we mean a polynomial in t with real coefficients and the coefficient of t^n is not equal to zero. From the last given condition we must have

(1) $$c_p u^{(p)} + c_{p-1} u^{(p-1)} + \dots + c_2 u'' + c_1 u' = 0$$

where $c_p \neq 0$, c's are real. From differential equations we know that (1) has polynomial solutions. If the solution is a polynomial of degree r and, (i) $r > p - 1$, it would imply that p of these derivatives are linearly independent; (ii) $r < p - 1$, it would imply that the first $(p - 1)$ of these derivatives are linearly dependent. Hence u is a polynomial of degree $p - 1$. More generally, we have

$$u = u(t) = \sum_{i=0}^{p-1}{}' \alpha_i \; v_i(t)$$

where $v_i(t)$ is a polynomial of degree i and they form a basis in $u(t)$. The fact u belongs to a fixed subspace of dimension p follows from the representation of u by its summands $(\alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_{p-1})$ and theorem 1 (P. R. Halmos, Finite Dimensional Vector Spaces, p. 29): The dimension of a direct sum is the sum of dimensions of it summands.

Also solved by Richard K. Guy, University of Malaya, Singapore and the proposer.

QUICKIES

From time to time this department will publish problems which may be solved by laborious methods, but which with the proper insight may be disposed of with dispatch. Readers are urged to submit their favorite problems of this type, together with the elegant solution and the source, if known.

Q170. Evaluate $\int_0^{\pi/2} \sin^2 x \, dx$ [*Submitted by R. F. Benton*].

Q171. A circular swimming pool has three equally spaced landing places, A, B and C. A swimmer starts from P on the edge and swims to A and back in 5 minutes, to B and back in 9 minutes. Assuming constant speed and no loss on turns how long will it take him to swim to C and back? [*Submitted by Richard K. Guy.*]

Q172. If the orthic triangle is similar to the basic triangle, what is the value of the ratio of this similitude? [*Submitted by N.A.Court.*]

Q173. In how many zeros does 10,000! end? [*Submitted by M.S.Klamkin*]

Q174. What is the total of the digits flashed on an electric clock during a regulation basketball game with no overtime periods? [*Submitted by V.C.Harris*]

Q175. A man is on a bridge from A to B, 3/8 of the way across from A. He hears a train approaching A at a rate of 60 miles per hour. If he runs toward A he will meet the train at A. If he runs twoard B the train will overtake him at B. How fast can the man run? [*Submitted by J.M.Howell*]

Q176. Sum $\displaystyle\sum_{n=0}^{\infty} \frac{n^4 - 6n^3 + 11n^2 - 6n + 1}{n!}$

[*Submitted by M.S.Klamkin*]

ANSWERS

A170. Note that $I = \displaystyle\int_0^{\pi/2} \sin^2 x\, dx = \int_0^{\pi/2} \cos^2 x\, dx.$

Hence $2I = \displaystyle\int_0^{\pi/2} dx,$ thus $I = \pi/4$

A171. By Ptolemy's Theorem the time will be 9 ± 5 minutes according as P is on the minor or major arc AB.

A172. The circumradii of the two triangles are corresponding elements in this similitude, and their ratio is known to equal ½.

A173. We have $\dfrac{10000}{5} + \dfrac{10000}{25} + \dfrac{10000}{125} + \dfrac{10000}{625} + \dfrac{10000}{5^5} + \cdots =$

$2000 + 400 + 80 + 16 + 3 = 2499$ zeros.

A174. The clock starts at 20:00 minutes and ends at zero for each half. A digit changes only when it is necessary for it to do so. During the first half, the minutes-ten's digit runs thru 2 and 1, once with total 3 and the minutes-unit's digit runs thru 9, 8, ..., 1 twice with total $45\cdot 2 = 90$. During twenty times with total $15 \cdot 20 = 300$ and the seconds-unit's digit runs thru 9, 8, ... 1 six times a minute for twenty minutes with a total $45 \cdot 6 \cdot 20 = 54000$.

The total for the game is $2\,[3 + 90 + 300 + 5400] = 11{,}586$.

A175. The man can run $5/8 - 3/8 = 1/4$ of the bridge in the time it takes the train to cross the bridge. Therefore he can run 1/4 (60 mph) or 15 mph.

A176.

$$\sum_{n=0}^{\infty} \frac{n^4 - 6n^3 + 11n^2 - 6n + 1}{n!} = \sum_{n=0}^{\infty} \frac{n(n-1)(n-2)(n-3) + 1}{n!} = 2\sum_{0}^{\infty} \frac{1}{n!} = 2e$$

Workshop For College Professors

The University of Michigan will offer its fourth annual *Workshop for College Professors* from June 25 to July 13, 1956. Features include presentations by a special workshop staff, discussions, and projects related to individual members' needs.

The *Workshop* will be directed by Algo D. Henderson, Professor of Higher Education, assisted by John E. Milholland, Assistant Professor of Psychology, and James M. Davis, Assistant Professor of Education and Director of the International Center, University of Michigan. Other University faculty will be available as resource persons, especially to assist individuals to develop new ideas and fresh materials for their academic courses.

Members of the special staff for the *Workshop* are Benjamin Bloom, University Examiner, University of Chicago; Frank R. Kille, Dean of the College, Carleton College; and Tremaine McDowell, Chairman, Program in American Studies, University of Minnesota. They will discuss such topics as course planning, teaching techniques and evaluation.

Additional information may be obtained by writing to the Director, Algo D. Henderson, 2442 U.E.S., University of Michigan, Ann Arbor, Michigan.

* * * * * * * * * * * * * * *

California Conference For Teachers Of Mathematics

Plans for the 1956 California Conference for Teachers of Mathematics have been completed on the Los Angeles campus of the University of California by the department of mathematics and University Extension. Co-operating groups are the California Mathematics Council and the National Council of Teachers of Mathematics.

Made up of two study groups and elementary and secondary laboratories, the conference will meet from June 20 through July 3. The University will grant one unit of credit for each laboratory or study group, with a maximum of three units for the entire conference, according to Clifford Bell, Professor of Mathematics and Head of Mathematics Extension, who is director of the conference. U.C.L.A. fees for the campus conference are $20 for each laboratory, or $30